Systems & Control Letters 58 (2009) 7–16

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

A continuous time linear adaptive source localization algorithm, robust to persistent drift

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ARTICLE INFO

Article history:
Received 9 May 2006
Received in revised form 18 June 2008
Accepted 17 July 2008
Available online 20 August 2008

Keywords:
Source localization
Adaptive
Continuous time
Estimation
Persistent excitation

ABSTRACT

The problem of source localization has assumed increased importance in recent years. In this paper, we formulate a continuous time adaptive localization algorithm, that permits a mobile agent to estimate the location of a stationary source, using only the measured distance from the source. The algorithm is shown to be exponentially asymptotically stable, under a persistent excitation condition that has an appealing interpretation. We quantify the fact that exponential asymptotic stability endows the algorithm with the ability to track slow, bounded but potentially persistent and nontrivial drift in the source.

1. Introduction

Over the last few years the problem of source localization has assumed increasing significance, [1]. It refers to an agent estimating the precise location of a source, using information related to the relative position of the agent and the source. This information can be of different kinds, for example distance, bearing, power level (which is indirectly related to distance) and time difference of arrival information, where two agents are involved. *In this paper, we will focus on distance estimates only.* Examples where source localization is important are many. Thus a base station in a cellular network may have to estimate the location of a phone in its region of coverage. In sensor networks, groups of sensors may have to estimate the location of an object or a node to facilitate routing, rescue, target tracking and proper network coverage.

We note that such distances can be estimated through two possible means. The first, passive measurement, involves a source that emits a signal and the signal intensity at the point of arrival at the agent location, together with characteristics of the propagation medium provides a distance estimate. Alternatively, in active distance measurements an agent transmits signals, and estimates the distance by measuring the time it takes for this signal reflected off the source to return.

There are two research thrusts in this area. In the first, clusters of stationary agents collaborate to localize a source. In two dimensions this would generically require that at least three distinct non-collinearly situated agents use their distances from the source they seek to localize. To be precise, with just two agents, the source position can be determined to within a binary ambiguity. Occasionally, *a priori* information may be available which will resolve that ambiguity. Otherwise, a third agent needs to be involved. In three dimensions, one generically needs at least four agents that do not lie on the same two dimensional plane. Several papers have proposed collaborative localization algorithms under a variety of assumptions concerning the manner in which distances are estimated, [2–6]. In the second thrust a single mobile agent, exploits its motion to localize, [7,8].

This paper is in the second category. It would seem relatively straightforward to achieve localization of a source with a mobile agent: one simply needs to take one distance measurement, move the agent, take another distance measurement, and then move it again to a position that is not collinear with the first two, and take a third measurement (and in three dimensions, a fourth measurement after another move). However, there are disadvantages to this, principally stemming from the fact that measurements are likely to be contaminated by noise, and from the fact that the source may move while the agent is moving to its new position. To address these concerns, we formulate a *continuous time algorithm*, that adaptively localizes a source through known agent motion, in three dimensions. As noted, the

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doi:10.1016/j.sysconle.2008.07.008

first motivation for adaptation, as opposed to a stand-alone batch algorithm stems from the need to achieve robust localization, given noisy measurements. A more sophisticated argument will be used to show that we can localize not only stationary sources, but also those that undergo slow, but possibly persistent movements, albeit with some error.

In Section 2 we formulate a continuous time algorithm that achieves exponentially fast localization of a stationary source, under a *persistent excitation* (p.e.) condition on the path of the moving agent. To understand the significance of this algorithm, we shall argue in Section 2 that in continuous time, distance measurements together with their derivatives, should suffice under the right conditions to secure source localization. However, while distance measurements are directly available, their derivatives are not. Nor is it desirable to perform explicit differentiation, as poor noise performance will ensue. The algorithm of this paper avoids such explicit differentiation, and thus localizes without noise amplification. In Section 3 we show that this algorithm can track a source experiencing slow, bounded but potentially persistent drift. In Section 4 we provide an intuitively appealing interpretation of the p.e. condition of Section 2. Section 5 provides simulation results, which among other things reveal a tradeoff between the ability to track a nonstationary source, with smooth noise performance. Section 6 is the conclusion. A conference version of this paper is [15].

2. The localization algorithm

We deal with a source positioned at a location whose coordinates in three dimensions are in a vector $x \in \mathbb{R}^3$. An agent whose coordinates at time t are in a vector $y(t) \in \mathbb{R}^3$, must estimate x from the knowledge of its own position over $t \in [0, \infty)$ and its distance from the source

$$d(t) = \|y(t) - x\|, \tag{2.1}$$

also over $t \in [0, \infty)$. Here, as in the rest of the paper, all norms refer to the 2-norm. In the sequel the following standing assumption will hold:

Assumption 2.1. The agent trajectory $y : \mathbb{R} \to \mathbb{R}^3$, is twice differentiable. Further, there exists $M_0 > 0$, such that

$$\forall t \in \mathbb{R}: \quad \|y(t)\| + \|\dot{y}(t)\| + \|\ddot{y}(t)\| \le M_0.$$

This assumption ensures that the motion of the agent can be executed with finite force.

Now observe that with $x \in \mathbb{R}^3$ under Assumption 2.1 and (2.1) one obtains,

$$\forall t \in \mathbb{R}: \quad \frac{\mathrm{d}}{\mathrm{d}t}\{d^2(t)\} = 2\dot{y}(t)^{\mathrm{T}}(y(t) - x). \tag{2.2}$$

Thus, if the derivatives of $d(\cdot)$ and $y(\cdot)$ be available over an interval $[0, T]$, and the velocity trajectory $\dot{y}(\cdot)$ span $\mathbb{R}^3$ over $[0, T]$, the sensor location x can be estimated. Pursuing such an approach in practice however, would require explicit differentiation of measured signals with accompanying noise amplification. To avoid such explicit differentiation, we invoke instead, the device of state variable filtering, popularly employed in the adaptive systems literature, [9]. A novelty of our approach lies in the fact that while in the adaptive systems literature state variable filtering has generally involved signals that are linearly related, in this context the underlying relationships are nonlinear.

Indeed we consider the signals $\eta(\cdot)$, $m(\cdot)$ and $V(\cdot)$, that are respectively the state variable filtered versions of $d^2(\cdot)/2$, $\|y(\cdot)\|^2/2$ and $y(\cdot)$. These are given in (2.3)–(2.8): For $\alpha > 0$ generate, under assumption (2.1)

$$\dot{z}_1(t) = -\alpha z_1(t) + \frac{1}{2}d^2(t), \quad z_1(0) = 0, \tag{2.3}$$

$$\eta(t) = -\alpha z_1(t) + \frac{1}{2}d^2(t), \tag{2.4}$$

$$\dot{z}_2(t) = -\alpha z_2(t) + \frac{1}{2}y^{\mathrm{T}}(t)y(t), \quad z_2(0) = 0, \tag{2.5}$$

$$m(t) = -\alpha z_2(t) + \frac{1}{2}y^{\mathrm{T}}(t)y(t), \tag{2.6}$$

$$\dot{z}_3(t) = -\alpha z_3(t) + y(t), \quad z_3(0) = 0, \tag{2.7}$$

and

$$V(t) = -\alpha z_3(t) + y(t). \tag{2.8}$$

Note that the generation of $\eta(t)$, $m(t)$ and $V(t)$ requires simply the measurements $d(t)$ and the knowledge of the localizing agent's own position, and can be performed without explicit differentiation.

In the sequel, we denote p as the derivative operator, i.e. $p \triangleq \mathrm{d}/\mathrm{d}t$. Then:

$$\frac{1}{p+\alpha}\left\{\frac{1}{2}d^2(\cdot)\right\} = \int_0^{\cdot} \mathrm{e}^{-\alpha(\cdot-\tau)}\frac{1}{2}d^2(\tau)\mathrm{d}\tau$$

just as

$$\frac{p}{p+\alpha}\left\{\frac{1}{2}d^2(\cdot)\right\} = \int_0^{\cdot} \mathrm{e}^{-\alpha(\cdot-\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{1}{2}d^2(\tau)\right\}\mathrm{d}\tau.$$

Further for two functions $a, b : \mathbb{R}_{\ge 0} \to \mathbb{R}^3$ we say $a(\cdot) \approx b(\cdot)$, if there exist $\lambda, M > 0$ such that for all $t \ge 0 \|a(t) - b(t)\| \le M\mathrm{e}^{-\lambda t}$. Then we have the following key Lemma:

Lemma 2.1. *Suppose Assumption 2.1 holds, $x \in \mathbb{R}^3$ is a constant, and $\eta(t)$, $m(t)$ and $V(t)$ are as defined in (2.3)–(2.8) with $\alpha > 0$. Then there holds:*

$$\eta(\cdot) \approx m(\cdot) - V^{\mathrm{T}}(\cdot)x. \tag{2.9}$$

Proof. An elementary calculation using (2.3) and (2.4), provides the relationship:

$$\dot{\eta}(t) + \alpha\eta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}d^2(t)\right\}.$$

Then as $\alpha > 0$, in operator notation,

$$\eta(\cdot) \approx \frac{p}{p+\alpha}\left\{\frac{1}{2}d^2(\cdot)\right\}. \tag{2.10}$$

Similarly,

$$m(\cdot) \approx \frac{p}{p+\alpha}\left\{\frac{1}{2}y^{\mathrm{T}}(\cdot)y(\cdot)\right\}, \tag{2.11}$$

and

$$V(\cdot) \approx \frac{p}{p+\alpha}\{y(\cdot)\}. \tag{2.12}$$

Then we establish the following key relationship between $\eta(\cdot)$, $m(\cdot)$, $V(\cdot)$ and x that exploits the fact that x is a constant.

$$\begin{aligned}\eta(\cdot) &\approx \frac{p}{p+\alpha}\left\{\frac{1}{2}d^2(\cdot)\right\} \approx \frac{1}{p+\alpha}\left\{\dot{y}^{\mathrm{T}}(\cdot)(y(\cdot)-x)\right\}\\ &\approx \frac{p}{p+\alpha}\left\{\frac{1}{2}y^{\mathrm{T}}(\cdot)y(\cdot)\right\} - \left(\frac{p}{p+\alpha}\left\{y^{\mathrm{T}}(\cdot)\right\}\right)x\\ &\approx m(\cdot) - V^{\mathrm{T}}(\cdot)x. \quad \blacksquare\end{aligned}$$

Observe that (2.9) mirrors (2.2), but involves only signals whose generation requires no explicit differentiation. We now present the adaptive localization algorithm: With $\hat{x}(\cdot)$ the estimate of x, choose for a scalar *adaptation gain*, $\gamma > 0$:

$$\dot{\hat{x}}(t) = -\gamma V(t)(\eta(t) - m(t) + V^{\mathrm{T}}(t)\hat{x}(t)). \tag{2.13}$$

Define

$$\tilde{x}(t) = \hat{x}(t) - x. \tag{2.14}$$

Then because of (2.9), (2.13) becomes:

$$\dot{\tilde{x}}(\cdot) \approx -\gamma V(\cdot)V^{\mathrm{T}}(\cdot)\tilde{x}(\cdot). \tag{2.15}$$

Localization will require that $\tilde{x}(\cdot)$ converge to zero. We then have the following theorem.

Theorem 2.1. *Suppose Assumption 2.1 holds and $x \in \mathbb{R}^3$ is a constant. Consider $\eta(t)$, $m(t)$ and $V(t)$ defined in (2.3)–(2.8), with $\alpha > 0$. Then under (2.15) there exist $\rho_1, \rho_2, \lambda > 0$ such that for all $t \geq 0$ and $\|x(0)\|$*

$$\|\tilde{x}(t)\| \leq (\rho_1\|x(0)\| + \rho_2)\,\mathrm{e}^{-\lambda t}$$

if and only if there exist $\alpha_1 > 0, \alpha_2 > 0, T > 0$ such that for all $t \geq 0$

$$\alpha_1 I \leq \int_t^{t+T} V(\tau)V^{\mathrm{T}}(\tau)\mathrm{d}\tau \leq \alpha_2 I. \tag{2.16}$$

Proof. It is well known, see e.g. [10], that the linear time varying system with $\gamma > 0$

$$\dot{z}(t) = -\gamma V(t)V^{\mathrm{T}}(t)z(t) \tag{2.17}$$

is exponentially asymptotically stable iff (2.16) holds. Hence the result follows. ■

The condition (2.16) is the celebrated p.e. condition. In Section 4 we interpret it to show that it is in accord with intuition. Note that it is well known in the adaptive systems literature [12] that exponential convergence imparts robustness to modest departures from idealizing assumptions. Certainly it will ameliorate the effects of noise perturbing the distance measurements. In the next section we show that in fact it also permits (2.13) to track slow, bounded though possibly large and persistent drift in the source location.

3. Tracking drift

In view of Assumption 2.1, there exist $M_i > 0$, such that for all $t \in \mathbb{R}$,

$$\|y(t)\| \leq M_1, \quad \|\dot{y}(t)\| \leq M_2 \quad \text{and} \quad \|\ddot{y}(t)\| \leq M_3. \tag{3.1}$$

The analysis in the previous section assumed a stationary source specifically by assuming that x, the source location, is a constant vector. The fact that under the p.e. condition (2.16) the localization algorithm is exponentially asymptotically stable, suggests the possibility of coping with departures from idealizing assumptions. One such departure of particular practical import is when the assumption of a stationary source is dropped. Rather, the source may experience slow, but persistent, drift that results in significant movement from its original position. One notes that should this drift eventually cease then the fact that under the conditions of Theorem 2.1 convergence to this terminal position will occur, is almost immediate. Instead we treat here the case that the source potentially never ceases to move.

Specifically, we make the following assumption on source motion.

Assumption 3.1. The source trajectory $x : \mathbb{R} \to \mathbb{R}^3$ is differentiable and there exist $M_4 > 0$ and $\epsilon > 0$, such that for all $t \in \mathbb{R}$

$$\|x(t)\| \leq M_4, \tag{3.2}$$

and

$$\|\dot{x}(t)\| \leq \epsilon. \tag{3.3}$$

In this section we will further assume that ϵ in (3.3) is "small". Observe that even though this assumption constrains the drift in the source to be bounded and slow, the net extent of the drift, i.e. the total change in $x(t)$ over a large time interval, is permitted to be substantial. We remark that the bound (3.2) is also reasonable. Under Assumption 2.1, $y(t)$ remains bounded for all time. If (3.2) failed, then the distance measurements from the agent to the source would be arbitrarily large, and possibly along asymptotically parallel lines. In the limit, this is like having collinear measurement points that fundamentally impair the ability to localize without ambiguity.

Since the stationarity assumption on the source has now been relaxed, (2.9) must also be accordingly modified. Indeed we have the following lemma.

Lemma 3.1. *Suppose Assumptions 2.1 and 3.1 hold, and $\eta(t)$, $m(t)$ and $V(t)$ are as defined in (2.3)–(2.8). Then there exists a signal $M_5 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for a suitable K_1 depending only on M_1, M_2, M_4 and α,*

$$|\eta(t) - m(t) + V^{\mathrm{T}}(t)x(t)| \leq M_5(t) \quad \forall t \geq 0, \tag{3.4}$$

and

$$M_5(\cdot) \approx K_1\epsilon. \tag{3.5}$$

Proof. Using the operator notation introduced in the proof of Lemma 2.1 one obtains,

$$\begin{aligned}\eta(\cdot) &\approx \frac{p}{p+\alpha}\left\{\frac{1}{2}d^2(\cdot)\right\} = \frac{1}{p+\alpha}\left\{(\dot{y}(\cdot)-\dot{x}(\cdot))^{\mathrm{T}}(y(\cdot)-x(\cdot))\right\}\\ &= \frac{p}{p+\alpha}\left\{\frac{1}{2}y^{\mathrm{T}}(\cdot)y(\cdot)\right\} - \left(\frac{1}{p+\alpha}\left\{\dot{y}^{\mathrm{T}}(\cdot)x(\cdot)\right\}\right) - f(\cdot)\\ &\approx m(\cdot) - \left(\frac{1}{p+\alpha}\left\{\dot{y}^{\mathrm{T}}(\cdot)x(\cdot)\right\}\right) - f(\cdot)\end{aligned} \tag{3.6}$$

where we have used (2.11) and the definition

$$f(\cdot) = \left(\frac{1}{p+\alpha}\left\{\dot{x}^{\mathrm{T}}(\cdot)(y(\cdot)-x(\cdot))\right\}\right). \tag{3.7}$$

Thus because of Assumptions 2.1 and 3.1, there exists an $F : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that for all $t \geq 0$,

$$|f(t)| \leq F(t) \tag{3.8}$$

and

$$F(.) \approx \frac{(M_1 + M_4)\epsilon}{\alpha}. \tag{3.9}$$

Focus now on the second term in (3.6). Observe now that

$$\frac{1}{p+\alpha}\left\{\dot{y}^{\mathrm{T}}(\cdot)x(\cdot)\right\} \approx Q(\cdot), \tag{3.10}$$

where with $C \in \mathbb{R}^3$,

$$\begin{aligned} Q(t) &= \mathrm{e}^{-\alpha t}\int_0^t \mathrm{e}^{\alpha\tau}\dot{y}^{\mathrm{T}}(\tau)x(\tau)\mathrm{d}\tau \\ &= \mathrm{e}^{-\alpha t}\left[\left(\int_0^{\tau}\mathrm{e}^{\alpha s}\dot{y}(s)\mathrm{d}s + C\right)^{\mathrm{T}} x(\tau)\right]_0^t \\ &\quad - \mathrm{e}^{-\alpha t}\int_0^t\left(\int_0^{\tau}\mathrm{e}^{\alpha s}\dot{y}(s)\mathrm{d}s + C\right)^{\mathrm{T}}\dot{x}(\tau)\mathrm{d}\tau \\ &= \left[\left(\int_0^{\tau}\mathrm{e}^{-\alpha(t-s)}\dot{y}(s)\mathrm{d}s + C\mathrm{e}^{-\alpha t}\right)^{\mathrm{T}} x(\tau)\right]_0^t - G(t), \end{aligned} \tag{3.11}$$

where

$$G(t) = \mathrm{e}^{-\alpha t}\int_0^t\left(\int_0^{\tau}\mathrm{e}^{\alpha s}\dot{y}(s)\mathrm{d}s + C\right)^{\mathrm{T}}\dot{x}(\tau)\mathrm{d}\tau. \tag{3.12}$$

Thus, as $\alpha > 0$

$$Q(\cdot) \approx V^{\mathrm{T}}(\cdot)x(\cdot) - G(\cdot). \tag{3.13}$$

Further from (3.12) one obtains.

$$|G(t)| \le \mathrm{e}^{-\alpha t}M_2\epsilon\left[\frac{\mathrm{e}^{\alpha t}-1}{\alpha^2} + t\left(\|C\| - \frac{1}{\alpha}\right)\right]. \tag{3.14}$$

Then the result follows from (3.4) to (3.14). ■

Then in view of Theorem 2.1 we have the following result.

Theorem 3.1. *Suppose Assumptions 2.1 and 3.1 hold, and there exist $\alpha_1, \alpha_2, T > 0$ such that for all $t \ge 0$ (2.16) holds. Consider $\eta(t)$, $m(t)$ and $V(t)$ defined in (2.3)–(2.8). Then $\hat{x}(t)$ given by (2.13) obeys for some K obtained from M_1, M_2, M_4, γ, α, T, α_1 and α_2, $\limsup_{t\to\infty}|\hat{x}(t) - x(t)| = K\epsilon$.*

Proof. Because of (2.13) and (2.14) there holds

$$\begin{aligned} \dot{\tilde{x}}(t) &= \dot{\hat{x}}(t) - \dot{x}(t) \\ &= -\gamma V(t)(\eta(t) - m(t) + V^{\mathrm{T}}(t)\hat{x}(t)) - \dot{x}(t) \\ &= -\gamma V(t)V^{\mathrm{T}}(t)\tilde{x}(t) - \gamma V(t)(\eta(t) - m(t) \\ &\quad + V^{\mathrm{T}}(t)x(t)) - \dot{x}(t) \\ &= -\gamma V(t)V^{\mathrm{T}}(t)\tilde{x}(t) + G_2(t), \end{aligned}$$

where

$$G_2(t) = -\gamma V(t)(\eta(t) - m(t) + V^{\mathrm{T}}(t)x(t)) - \dot{x}(t).$$

Then because of Lemma 3.1, (3.3) and the fact that $V(\cdot)$ is bounded, There exists a $K_5 > 0$ obtained from M_1, M_2, M_4, γ and α, and an $M_6 : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, obeying $M_6(\cdot) \approx K_5\epsilon$ such that $|G_2(t)| \le M_6(t)\forall t \ge 0$. Hence the result follows from the exponential asymptotic stability of (2.17). ■

We note that K in Theorem 3.1 itself depends on the convergence rate of (2.17), which according to [11] increases linearly with α_1 and declines quadratically with α_2 and linearly with T. Specifically, with γ as in (2.13), $z(\cdot)$ as in (2.17) and α_i as in (2.16), for all t, there holds

$$\frac{\|z(t+T)\|^2 - \|z(t)\|^2}{\|z(t)\|^2} \le \frac{4\gamma\alpha_1}{1+\gamma\alpha_2+\gamma^2\alpha_2^2}.$$

What Theorem 3.1 shows is that under the p.e. condition in (2.16), one can track sustained but bounded drift in the source provided the drift is sufficiently slow, underscoring the robustness of the proposed localization algorithm. In the next section we provide a physical interpretation of this p.e. condition.

4. Persistent excitation

In this section, we explore the meaning of the p.e. condition in (2.16). First, observe that in three dimensions, an agent cannot generically localize without ambiguity if its motion is confined to a plane, as there would be at least two points separated by this plane that may provide the same distance measurements. An exception arises when there is an additional level of nongenericity in that not only is the motion of the agent confined to a plane, but the source lies in the same plane. Similarly in two dimensions generically the motion cannot be exclusively collinear. Since the source location is unknown; for all practical purposes planar agent motion in three dimensions and collinear motion in two dimensions should be avoided. In this section we quantify the relationship between avoiding planar motion and satisfying (2.16).

To this end, we first relate the p.e. condition on $V(\cdot)$ to one on $\dot{y}(\cdot)$, by exploiting techniques used to establish transfer of excitation conditions developed in [13] for adaptive identification and control problems. From (2.7) and (2.8), we have

$$\dot{V}(t) + \alpha V(t) = \dot{y}(t). \tag{4.1}$$

We are interested in showing that (2.16) holds iff a p.e. condition holds on $\dot{y}(\cdot)$. The transfer of excitation results of [13] do not directly apply to this setting, as they involve scalar inputs, and furthermore require that the system relating the two signals be proper, which the V to $\dot{y}$ system is not.

To prove this result we invoke the following specialization of an inequality from [14], which has been used before in establishing transfer of excitation relations in [13].

Lemma 4.1. *Suppose on a closed interval $\mathcal{I} \subset \mathbb{R}$ of length Δ, a signal $w : \mathcal{I} \to \mathbb{R}$ is twice differentiable, and for some ϵ and M'*

$$|w(t)| \le \epsilon_1 \quad \text{and} \quad |\ddot{w}(t)| \le M' \quad \forall t \in \mathcal{I}.$$

Then for some M independent of ϵ_1, $\mathcal{I}$ and M', and $M'' = \max(M', 2\epsilon_1\Delta^{-2})$ one has:

$$|\dot{w}(t)| \le M(M''\epsilon_1)^{1/2} \quad \forall t \in \mathcal{I}.$$

The upper bound of (2.16) holds because of Assumption 2.1 and the stability of the state variable filters. We will thus focus on the lower bound of (2.16). To this end we have the following theorem.

Theorem 4.1. *Suppose Assumption 2.1 and (4.1) hold with arbitrary $V(0) \in \mathbb{R}^3$. Then there exist $\alpha_1, T > 0$, such that the lower bound in (2.16) holds for all $t \ge 0$ if and only if there exist $\beta_1, \bar{T} > 0$ such that for all $t \ge 0$*

$$\beta_1 I \le \int_t^{t+\bar{T}} \dot{y}(\tau)\dot{y}(\tau)^{\mathrm{T}}\mathrm{d}\tau. \tag{4.2}$$

Proof. We will prove that the violation of one lower bound is equivalent to the violation of the other.

Suppose the lower bound in (4.2) is violated. Then for all $\epsilon_2 > 0$ and $T > 0$, there exists a t_0 and unit norm $\theta \in \mathbb{R}^3$, such that

$$\int_{t_0}^{t_0+T}(\theta^{\mathrm{T}}\dot{y}(\tau))^2\mathrm{d}\tau \le \epsilon_2^2.$$

Thus from Lemma 4.1 for some M_7, all $\epsilon_2 > 0$, some $T_1(\epsilon_2)$, dependent only on the bound on $\ddot{y}(\cdot)$ and ϵ_2, and all $T > T_1(\epsilon_2)$, there exists a t_0 and unit norm $\theta \in \mathbb{R}^3$, for which

$$\left|\theta^{\mathrm{T}}\dot{y}(t)\right| \le M_7\epsilon_2^{1/2} \quad \forall t \in [t_0, t_0+T].$$

Thus because of (4.1),

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\{\theta^{\mathrm{T}}V(t)\}+\alpha\theta^{\mathrm{T}}V(t)\right|\leq M_7\epsilon_2^{1/2}\quad\forall t\in[t_0,t_0+T].$$

As $\alpha > 0$ in (4.1), for every $\epsilon_2 > 0$, some K_6 dependent only on M_7 and α, and a $T_2(\epsilon_2)$ dependent on M_7, α and ϵ_2, and all $T > \max\{T_1(\epsilon_2), T_2(\epsilon_2)\}$,

$$\left|\theta^{\mathrm{T}}V(t)\right|\leq K_6\epsilon_2^{1/2},\quad\forall t\in[t_0+T_2(\epsilon_2),t_0+T].$$

Since this holds for every $\epsilon_2 > 0$, there will exist a value of ϵ_2 such that $|\theta^{\mathrm{T}}\dot{y}(t)\dot{y}^{\mathrm{T}}(t)\theta| < \frac{\beta_1}{T}$ for all $t \geq t_0$, which violates the lower bound of (2.16). Hence the satisfaction of the constraint of (2.16) ensures that (4.2) holds.

Now suppose that the lower bound in (2.16) is violated. Then for all $\epsilon_3 > 0$ and $T > 0$, there exists a t_0 and unit norm $\theta \in \mathbb{R}^3$, such that

$$\int_{t_0}^{t_0+T}(\theta^{\mathrm{T}}V(\tau))^2\mathrm{d}\tau\leq\epsilon_3^2.$$

Thus from Lemma 4.1 for some M_8, all $\epsilon_3 > 0$, some $T_3(\epsilon_3)$, dependent only on the bound on $\dot{V}(\cdot)$ and ϵ_3, and all $T > T_3(\epsilon_3)$, there exists a t_0 and unit norm $\theta \in \mathbb{R}^3$, for which

$$\left|\theta^{\mathrm{T}}V(t)\right|\leq M_8\epsilon_3^{1/2},\quad\forall t\in[t_0,t_0+T].$$

As $\ddot{y}(\cdot)$ is bounded, so is $\ddot{V}(\cdot)$. Thus again from Lemma 4.1 for some L, all $\epsilon_3 > 0$, some $T_4(\epsilon_3)$, dependent only on the bounds on $\dot{V}(\cdot)$, $\ddot{V}(\cdot)$ and ϵ, and all $T > T_4(\epsilon_3)$, there exists a t_0 and unit norm $\theta \in \mathbb{R}^3$, for which

$$\left|\theta^{\mathrm{T}}\dot{V}(t)\right|\leq L\epsilon_3^{1/4},\quad\forall t\in[t_0,t_0+T].$$

Consequently, because of (4.1)

$$\left|\theta^{\mathrm{T}}\dot{y}(t)\right|\leq\alpha M_8\epsilon_3^{1/2}+L\epsilon^{1/4},\quad\forall t\in[t_0,t_0+T].$$

As before this violates the lower bound of (4.2). ■

Let us now tie (4.2) to the avoidance of planar motion. Observe that should in particular the Gramian in (4.2) be singular then, because of Assumption 2.1 for some unit norm θ, $\theta^{\mathrm{T}}\dot{y}(t) = 0$ for all t. This in particular means that for this constant unit norm $\theta \in \mathbb{R}^3$, there exists a $C_1 \in \mathbb{R}^3$ such that for all t, $\theta^{\mathrm{T}}y(t) = C_1$ defining a planar motion. The p.e. condition in (4.2) effectively requires that the localizing agent avoid in some sense planar motion. In particular we have the following tangible connection that among other things allows the verification of the p.e. condition from $y(t)$ alone.

Theorem 4.2. *Suppose Assumption 2.1 and (4.1) hold with arbitrary $V(0) \in \mathbb{R}^3$. Then there exist $\alpha_1, T > 0$, such that the lower bound in (2.16) holds for all $t \geq 0$ iff there exist $\bar{T} > 0$ and $\beta > 0$ such that for all t, there exist $\{t_1, \ldots, t_4\} \in [t, t+\bar{T}]$, for which*

$$|\det([y(t_2)-y(t_1),y(t_3)-y(t_1),y(t_4)-y(t_1)])|>\beta. \tag{4.3}$$

Proof. Since $y(\cdot)$ is bounded because of Assumption 2.1, (4.3) is equivalent to the requirement that for some $\beta_4 > 0$ the smallest eigenvalue of

$$\begin{aligned}&[y(t_2)-y(t_1),y(t_3)-y(t_1),y(t_4)-y(t_1)]\\&\times[y(t_2)-y(t_1),y(t_3)-y(t_1),y(t_4)-y(t_1)]^{\mathrm{T}}\end{aligned}$$

must be greater than β_4. Thus (4.3) is is equivalent to the requirement that for some $\beta_4 > 0$

$$\sum_{i=2}^{4}(y(t_i)-y(t_1))(y(t_i)-y(t_1))^{\mathrm{T}}>\beta_4 I. \tag{4.4}$$

In view of Theorem 4.1, we will focus on proving the equivalence of (4.4) and the lower bound in the (4.2) condition. To prove that the lower bound of (4.2) implies that of (4.4) we will show that the violation of (4.4) implies the violation of the lower bound of (4.2).

To this end we first assert that the violation of (4.4) in turn implies the following: that for some ϵ^*, and every $\epsilon^* > \epsilon_4 > 0$ and $T > 0$, there exists a $t_0 > 0$ and a unit norm $\theta \in \mathbb{R}^3$, such that

$$|\theta^{\mathrm{T}}(y(t)-y(t_0))|\leq\epsilon_4\quad\forall t\in[t_0,t_0+T]. \tag{4.5}$$

A formal proof of this result is involved, and is omitted. Briefly, it reflects the fact that if on a compact interval of time, all quartet of samples of $y(t)$ in $\mathbb{R}^3$ are close to being coplanar, then the entire trajectory is close to the same plane on this interval.

Thus from Lemma 4.1 for some M_9, ϵ^* all $\epsilon^* > \epsilon_4 > 0$, some $T_1(\epsilon_4)$, dependent only on the bound on $\ddot{y}(t)$ and ϵ_4, and all $T > T_3(\epsilon_4)$, there exists a t_0 and unit norm $\theta \in \mathbb{R}^3$, for which

$$|\theta^{\mathrm{T}}\dot{y}(t)|\leq M_9\epsilon_4^{1/2}\quad\forall t\in[t_0,t_0+T]$$

violating the lower bound of (4.2). Now suppose the lower bound of (4.2) is violated. Then for every $\epsilon_8 > 0$ and $T > 0$, there exists a t_0 and a unit norm $\theta \in \mathbb{R}^3$, such that

$$\int_{t_0}^{t}(\theta^{\mathrm{T}}\dot{y}(\tau))^2\mathrm{d}\tau\leq\epsilon_8^2\quad\forall t\in[t_0,t_0+T].$$

Thus from Lemma 4.1 for some M_{10}, all $\epsilon_8 > 0$, some $T_4(\epsilon_8)$, dependent only on the bound on $\ddot{y}(\cdot)$ and ϵ_8, and all $T > T_4(\epsilon_8)$, there exists a t_0 and unit norm $\theta \in \mathbb{R}^3$, for which

$$|\theta^{\mathrm{T}}\dot{y}(t)|\leq M_{10}\epsilon_8^{1/2}\quad\forall t\in[t_0,t_0+T].$$

Now suppose (4.4) does hold with some finite $\bar{T}$. This leads to a contradiction as for any $t_1 \geq t_0 + T_4(\epsilon_8)$ and $t_1 \leq t \leq t_1 + \bar{T}$, one has

$$|\theta^{\mathrm{T}}(y(t)-y(t_1))|\leq M_{10}\epsilon_8^{1/2}\bar{T}.\quad■ \tag{4.6}$$

Note that the determinant in (4.3) being zero, implies that $y(t_i)$ are coplanar. The parameter β in (4.3) measures how close they are to be situated on a plane. In effect this result shows that the p.e. condition can be verified by checking whether on each interval of a fixed length, there are at least four time points at which the agent positions are sufficiently removed from any single plane, specifically by testing (4.3).

We now provide an informal analysis of what happens when (2.16) is violated in the nongeneric situation, where the source location is coplanar with the agent trajectory. Suppose in particular that $y(\cdot)$ lies on the same plane for all $t \geq 0$, but avoids sufficiently often and to a sufficient extent a collinear path. Suppose also that for some scalar $cV(0) = c\dot{y}(0)$. Then of course for a scalar $a : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, uniformly bounded away from zero, $V(t) = a(t)\dot{y}(t)$. for all t. Consequently, even though the Gramian in (2.16) is singular, its projection on the two dimensional plane in which $y(\cdot)$ resides is uniformly positive definite. Now suppose $\hat{x}(0)$ also resides on this plane and the exponentially decaying terms in (2.9) are all zero. Then because of (2.13), the localization error $\tilde{x}(\cdot)$ is confined to this plane for all $t \geq t_0$, and cannot be orthogonal to the Gramian in (2.16). Consequently, a simple projection argument combined with the technique of [10] will show that localization will ensue at an exponential rate. A similar argument can be advanced when the agent executes a collinear motion along a line containing the source. This by no means contradicts our earlier assertion that the p.e. condition (2.16) is necessary for exponential asymptotic localization, as there is at least one set of initial conditions, namely involving initial errors that are orthogonal to the Gramian in

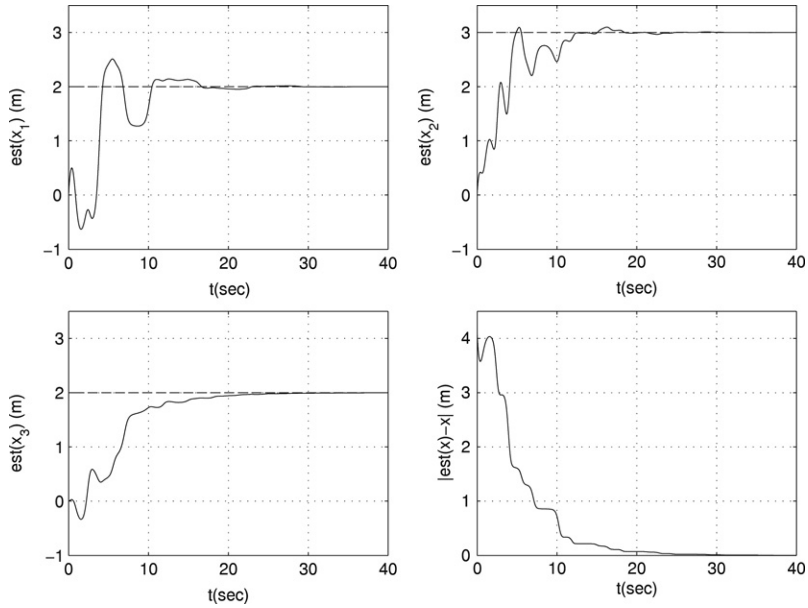

Fig. 1. Location estimation for $x = [2, 3, 2]^{\mathrm{T}}$ (m), $y(t) = [2 + 2\sin t,\ 2\cos 2t,\ 2\sin 0.5t]^{\mathrm{T}}$ (m), $\alpha = 1$. The dashed lines correspond to the actual coordinates of the source and the solid curves show the estimate trajectories.

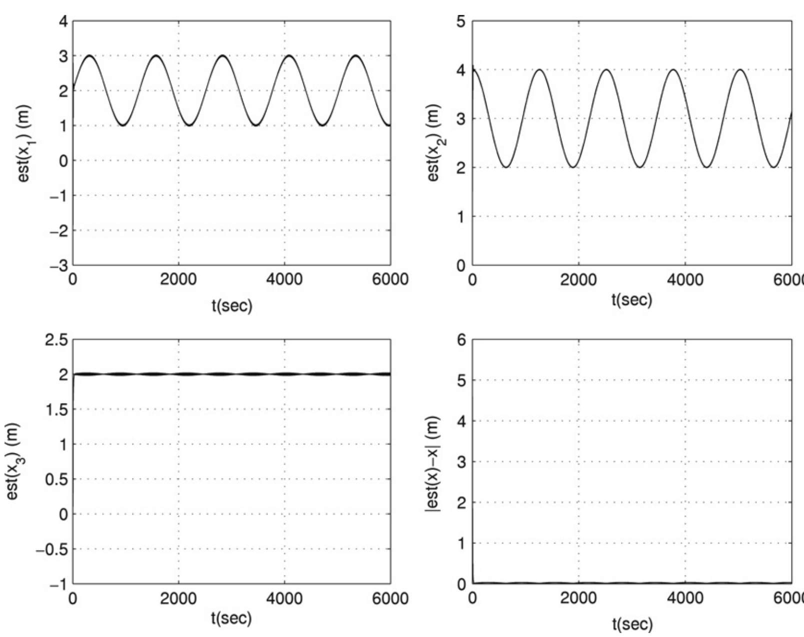

Fig. 2. Location estimation for $x(t) = [2 + \sin 0.005t,\ 3 + \cos 0.005t,\ 2]^{\mathrm{T}}$ (m), $y(t) = [2 + 2\sin t,\ 2\cos 2t,\ 2\sin 0.5t]^{\mathrm{T}}$ (m), $\alpha = 1$. The dashed curves correspond to the actual coordinates of the source and the solid curves show the estimate trajectories. Time scale 0–6000 s.

(2.16), which would correspond to false stationary points of the algorithm.

Indeed suppose that the agent executes a planar, though persistent noncollinear motion, and x is coplanar with $y(\cdot)$ for all $t \geq 0$. Then there exists an *orthogonal* matrix $A \in \mathbb{R}^{3\times 3}$ for which for all $t \geq 0$

$$Ax = \begin{bmatrix}0, & \bar{x}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}, \qquad Ay(t) = \begin{bmatrix}0, & \bar{y}^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}}, \quad \text{and}$$

$$A\dot{y}(t) = \begin{bmatrix}0, & \dot{\bar{y}}^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}},$$

and there exist $\beta_1, \beta_2, \bar{T} > 0$ such that for all $t \geq 0$

$$0 < \beta_1 I \leq \int_t^{t+\bar{T}} \dot{\bar{y}}(\tau)\dot{\bar{y}}(\tau)^{\mathrm{T}} \mathrm{d}\tau \leq \beta_2 I. \tag{4.7}$$

Here $\bar{x} \in \mathbb{R}^2$ and $\bar{y} : \mathbb{R} \to \mathbb{R}^2$. Observe, in this case with $\bar{V} : \mathbb{R} \to \mathbb{R}^2$,

$$AV(\cdot) \approx \begin{bmatrix}0, & \bar{V}^{\mathrm{T}}(\cdot)\end{bmatrix}^{\mathrm{T}},$$

and there exist $\alpha_1, \alpha_2, T > 0$ such that for all $t \geq 0$

$$0 < \alpha_1 I \leq \int_t^{t+T} \bar{V}(\tau)\bar{V}(\tau)^{\mathrm{T}} \mathrm{d}\tau \leq \alpha_2 I. \tag{4.8}$$

The orthogonality of A ensures that with $\tilde{\bar{x}} : \mathbb{R} \to \mathbb{R}^2$, and

$$A\tilde{x}(t) = \begin{bmatrix}\tilde{x}_1^{\mathrm{T}}(t), & \tilde{\bar{x}}^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}},$$

(2.15) reduces to

$$\begin{bmatrix}\dot{\tilde{x}}_1(\cdot) \\ \dot{\tilde{\bar{x}}}(\cdot)\end{bmatrix} \approx -\gamma AV(\cdot)V^{\mathrm{T}}(\cdot)A^{\mathrm{T}} \begin{bmatrix}\tilde{x}_1(\cdot) \\ \tilde{\bar{x}}(\cdot)\end{bmatrix}$$

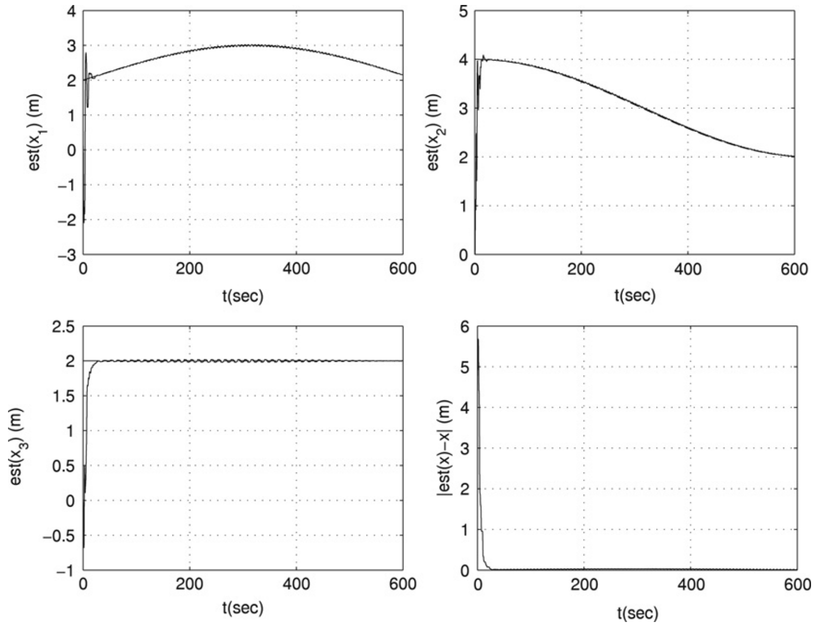

Fig. 3. Location estimation for $x(t) = [2 + \sin 0.005t,\ 3 + \cos 0.005t,\ 2]^{\mathrm{T}}$ (m), $y(t) = [2 + 2\sin t,\ 2\cos 2t,\ 2\sin 0.5t]^{\mathrm{T}}$ (m), $\alpha = 1$. The dashed curves correspond to the actual coordinates of the source, and the solid curves show the estimate trajectories. Time scale 0–600 s.

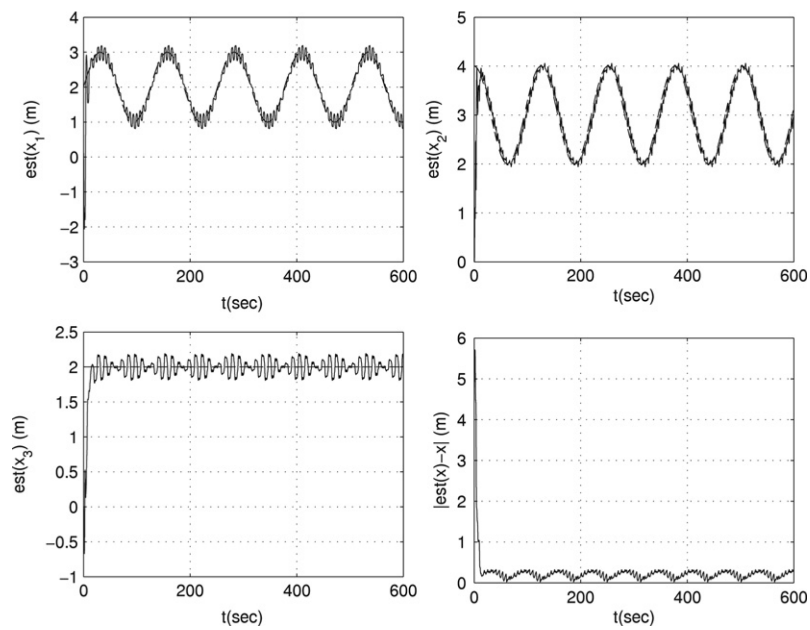

Fig. 4. Location estimation for $x(t) = [2 + \sin 0.05t,\ 3 + \cos 0.05t,\ 2]^{\mathrm{T}}$ (m), $y(t) = [2 + 2\sin t,\ 2\cos 2t,\ 2\sin 0.5t]^{\mathrm{T}}$ (m), $\alpha = 1$. The dashed curves correspond to the actual coordinates of the source, and the solid curves show the estimate trajectories. Time scale 0–600 s.

$$\approx\ -\gamma \begin{bmatrix} 0 & 0 \\ 0 & \bar{V}(\cdot)\bar{V}^{\mathrm{T}}(\cdot) \end{bmatrix} \begin{bmatrix} \tilde{x}_1(\cdot) \\ \bar{\tilde{x}}(\cdot) \end{bmatrix}. \tag{4.9}$$

Should $\tilde{x}_1(0) = 0$, i.e. the original estimate itself lie on the same plane as the source and the agent, *and one has that* $\dot{x}_1(\cdot) = 0$, *rather than* $\dot{x}_1(\cdot) \approx 0$, then one will have the exponential convergence of $\hat{x}(\cdot)$ to x. On the other hand even if $\tilde{x}_1(0) = 0$, and $\dot{x}_1(\cdot)$ is exponentially decaying, convergence cannot be guaranteed as $\tilde{x}_1(\cdot)$, may not converge to zero. Thus, generically, convergence is only possible if the agent knows *a priori* that it is coplanar with the source, and the state variable filters are initialized just the right way. Similar conclusions can be drawn for collinear motion of an agent collinear with the source.

5. Simulation results

In this section, we provide simulation results to demonstrate the performance of the localization algorithm in Section 2. In these examples, unless otherwise stated, the adaptation gain $\gamma = 1$.

First, consider a stationary source located at $x = [2, 3, 2]^{\mathrm{T}}$ (m). Assume that for all $t \geq 0$, $y(t) = [2 + 2\sin t,\ 2\cos 2t,\ 2\sin 0.5t]^{\mathrm{T}}$ (m). Assume that the filter pole in (2.3)–(2.8) is $\alpha = 1$. Then, using the localization algorithm (2.13), we obtain the source localization results shown in Fig. 1. As can be seen in Fig. 1, the source location estimation $\mathrm{est}(x) = \hat{x}$ converges to its actual value x exponentially fast.

Next, we consider the case where the source is moving around a nominal location at $x = [2, 3, 2]^{\mathrm{T}}$ (m) in particular for all $t \geq 0$, $x(t) = [2 + \sin 0.005t,\ 3 + \cos 0.005t,\ 2]^{\mathrm{T}}$ (m). Observe

Fig. 5. Location estimation for $x = [2, 3, 2]^{\mathrm{T}}$ (m), $y(t) = [2 + 2\sin t,\ 2\cos 2t,\ 2\sin 0.5t]^{\mathrm{T}}$ (m), $\alpha = 1$. Noise in distance measurement with power .001 (m^2). The dashed lines correspond to the actual coordinates of the source, and the solid curves show the estimate trajectories.

Fig. 6. Location estimation for $x = [2, 3, 2]^{\mathrm{T}}$ (m), $y(t) = [2 + 2\sin t,\ 2\cos 2t,\ 2\sin 0.5t]^{\mathrm{T}}$ (m), $\alpha = 1$. Noise in distance measurement with power .005 (m^2). The dashed lines correspond to the actual coordinates of the source, and the solid curves show the estimate trajectories.

that the net extent of the drift in the first two coordinates is 2, and thus substantial. The rate of drift (i.e ϵ in Assumption 3.1), on the other hand, is relatively small having an amplitude of 0.005. We expect from Theorem 3.1 that the agent will track the source movement with an error proportional to 0.005. Assume again that for all $t \geq 0$, $y(t) = [2 + 2\sin t,\ 2\cos 2t,\ 2\sin 0.5t]^{\mathrm{T}}$ (m). Using the localization algorithm (2.13) with the same state variable filter pole $\alpha = 1$, we obtain the results shown in Fig. 2, while Fig. 3 provides a snapshot of the intial tracking. The estimate $\hat{x}(t)$ tracks the motion of $x(t)$ with barely discernible error. Fig. 4 demonstrates tracking when the rate of drift is ten times that in Fig. 2, i.e. when for all $t \geq 0$, $x(t) = [2 + \sin 0.05t,\ 3 + \cos 0.05t,\ 2]^{\mathrm{T}}$ (m). As expected, the tracking though still quite good, is with a correspondingly larger error.

Now consider noise performance, specifically when the distance measurements $d(t)$ are perturbed by a zero mean bandlimited white Gaussian noise. Figs. 5–7 consider the stationary source of the first example, and noise variance of 0.001 (m^2), 0.005 (m^2) and 0.1 (m^2), respectively. While the first two are with unity adaptation gains, the last is with a much reduced adaptation gain of 0.01. We note that a noise variance of .001 (m^2) corresponds to an average noise amplitude of about 0.03 (m), which given the scale of the problem in terms of the actual distances involved, is quite reasonable. As expected in these two examples the tracking error scales with the noise magnitude.

Fig. 7 reveals the role of the adaptation gain. Specifically, in this example involving a very substantial noise, one finds that by significantly reducing the adaptation gain, in this case to 0.01, one significantly reduces the effect of noise in the long term tracking error. The price is the much reduced convergence speed, which of course impairs the ability to track source movements.