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Distributed control for optimal economic dispatch of a network of heterogeneous power generators

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Abstract—**In this paper, we present a simple, distributed algorithm for frequency control and optimal economic dispatch of power generators. In this algorithm each generator independently adjusts its power-frequency set-points of generators to correct for generation and load fluctuations using only the aggregate power imbalance in the network, which can be observed by each generator through local measurements of the frequency deviation on the grid. In the absence of power losses, we prove that the distributed algorithm eventually achieves optimality i.e. minimum cost power allocations, under mild assumptions (strict convexity and positivity of cost functions); we also present numerical results from simulations to compare its performance with traditional (centralized) dispatch algorithms. Furthermore, we show that the performance of the algorithm is robust in the sense that even with power losses it corrects for frequency deviations, and for low levels of losses, it still achieves near-optimal allocations; we present an approximate analysis to quantify the resulting suboptimality.**

I. Introduction

We present a simple distributed algorithm for load frequency control and economic dispatch in an electric grid; in this algorithm each generator uses local knowledge of its own cost of generation along with measurements of frequency deviations to dynamically adjust its real power generation. We show that over time in the course of responding to normal power fluctuations on the grid, the algorithm eventually achieves the condition of optimal economic dispatch under mild assumptions on the cost functions of the generators when there are no line losses (and near-optimal allocations as long as the line losses are not too large). This generalizes our earlier work in [1] where optimality was achieved only when all generators had the same underlying production cost i.e. identical cost functions.

A. Motivation

This work is motivated by the anticipated needs of the next generation electric grid which is expected to have smart consumer end-nodes [2] and a high penetration of alternative energy generators. Since the availability of alternative energy sources such as wind and solar generators is inherently intermittent in time and dispersed in geography [3], the ability of the electric grid to dynamically adjust generation and consumption is key to achieving a high load factor and efficient energy use. Decentralized control techniques are well-suited to such a flexible electric grid, and this recognition has led to an increased interest in concepts such as microgrids [4] and distributed generation (DG) [5]. Similarly, decentralized control techniques are extremely attractive to "smart grids" [6], where there is the possiblity of controlling loads in addition to generation, in response to real-time surplus or scarcity of power in the electric grid.

In a traditional electric grid, control of generators, i.e. Automatic Generation Control (AGC)[1] is accomplished on multiple time-scales using multiple different mechanisms [9]. Primary control is implemented in a distributed fashion at the generators, but secondary and tertiary control (corresponding to load frequency control (LFC) and economic dispatch (ED) respectively) are implemented from a centralized control station at the Load Serving Entity (LSE) and transmission system operator (TSO) [10]. The goal of the secondary control process is to reduce the Area Control Error (ACE) to zero. The ACE is a measure of the imbalance between rated generation capacity and power consumed within the control area, and the LFC algorithm adjusts the power generation levels in order to achieve power balance within the control area.

Traditionally, an *ad hoc* allocation is used by the secondary controller to return ACE to zero without consideration of cost minimization; the latter function is the responsibility of the tertiary control process or economic dispatch (ED). The economic dispatch process periodically re-allocates the total power among generators to minimize total cost. Once set by the dispatch algorithm, the power allocations may deviate over time from their optimal values because of cumulative load fluctuations and the actions of the secondary controller. The dispatch problem is typically formulated as a multivariable constrained optimization problem [11] that is then solved using Lagrangian techniques such as "lambda iteration" [12]. However, when line losses are included in the model, the dispatch problem becomes analytically intractable even with simplified models for the generator cost functions. In previous academic work on the dispatch problem, complex numerical optimization methods such as genetic algorithms, particle swarm optimization or Monte-Carlo methods [13], [14] are often employed to determine the minimum cost allocation of power across generators. In contrast, in the algorithm described in this paper, there is no centralized dispatcher;

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[1]Terms such as AGC [7], [8] may be used to mean slightly different things in the technical literature from different parts of the world, partly mirroring differences in the structure of the electric grid itself between Europe, North America and so on. In this paper, AGC is used as a generic term to include the hierarchy of mechanisms for frequency regulation, tie-line power flow control and economic dispatch.

instead each generator adjusts its own generation power iteratively and independently from other generators and this procedure is shown (under some conditions) to quickly correct for frequency deviations, and also eventually to converge to minimum cost allocations automatically. Thus this iterative procedure simultaneously performs the traditional functions of load frequency control and economic dispatch i.e. secondary and tertiary control.

The idea of using frequency deviations to control the power imbalance between load and generation has been considered before. As far back as 1980, the authors in [15] used the biological metaphor of homeostatis to motivate a power system where loads and generators dynamically adjust their power flows to maintain equilibrium. For this purpose, it was proposed that the system frequency could be used on short time-scales and a market-based pricing scheme along with the necessary metering and billing infrastructure for longer time-scales. It is remarkable that this paper presciently anticipates market-based pricing [16], [17], smart metering and other major subsequent developments in the electric grid.

The dispatch algorithm described in this paper is very much compatible with the vision outlined in [15]; we develop the idea suggested in [15] of using frequency for power balancing into a concrete, fully distributed algorithm and show that it achieves minimum cost allocations under certain conditions. In the decades since 1980, the electric grid has undergone significant changes such as deregulation, market-based pricing, increasing proportion of alternative energy sources and smart metering infrastructure, all of which provide opportunities for further developing this vision of a grid that adapts in a dynamic and distributed fashion to achieve high energy efficiency.

The dispatch algorithm proposed in this paper can also be thought of as an iterative Lagrangian optimization procedure (see e.g. [18], Chap. 4). We explore this connection further in Section IV.

B. Economic Dispatch and Electricity Markets

In recent decades, the electric power industry in the US (and in other countries) has undergone a significant amount of deregulation [19], [20]. The effect of this deregulation is to introduce market-based competition to the electric grid whereby different energy suppliers compete on price to provide power to the utility; this is in contrast to the previous model where a single vertically integrated utility company controlled generation, transmission and distribution of electric power. Under the new deregulated model, independent power producers are assigned generation schedules based on the outcome of a bidding process on one or more wholesale electricity markets (usually at least two markets on different time-scales, a "real-time" and a "day ahead" market [21]), and in this sense the generation schedule is determined in a "decentralized" way.

Note, however, that the market process is decentralized at the level of economic entities (i.e. generation companies and independent power producers), not on the level of individual generators. Each generation company still faces the engineering problem of optimally allocating the total generation among the individual generators. Better dispatch algorithms can be profitably used by power producers to make more competitive bids and maximize their returns in the wholesale market [22]. The methods discussed in this paper are thus complementary to the market-based pricing process.

Recent studies [23] have shown that Centralized Security Constrained Economic Dispatch (SCED) methods [24] are effective for managing conventional generators in the deregulated environment while taking into account congestion management and market operations in large-scale grids. *The distributed approach presented in this paper is not intended as an alternative to these methods.* However, it offers some attractive features that make it potentially useful for applications such as a grid with a significant number of small alternative energy generators.

C. Contributions

Our proposed algorithm is based on the following simple idea. If we neglect power losses, the minimum cost allocation of power is achieved when the marginal cost of an additional unit of generation is equal for all generators. Thus, when there is a positive power imbalance (i.e. instantaneous load exceeds rated generation), it is intuitively reasonable for a generator with a lower marginal cost to increase its generation by a larger amount than one with a higher marginal cost. Conversely, when there is a negative power imbalance, it is reasonable for a high marginal cost generator to reduce its generation by a relatively larger amount. On the other hand, one must also account for the fact that the generator whose cost function has a higher second derivative, will undergo a faster rise in its marginal cost for the same unit of added generation. The algorithm of this paper accordingly modifies the algorithm of [1] to accommodate this effect. As in [1], each generator has access only to its own cost function, and (local) measurements of frequency deviations which serves as an indirect measure of the power imbalance in the grid.

Our distributed approach offers the following features that makes it an interesting alternative to the traditional centralized approach for certain applications.

1) *Scalability.* Centralized dispatch algorithms require knowledge of the cost functions of each generator which limits their scalability. Our algorithm is fully distributed and thus, more scalable which makes it especially attractive for power grids supplied by a large number of small distributed generators.
2) *Dynamic adaptability.* The distributed algorithm responds automatically to changes in loads and in generation costs and modifies the power allocations accordingly. This can be attractive when the generation and loads are highly variable as in grids with a large number of intermittent alternative energy generators.
3) *Model independence.* The distributed approach solves the optimization problem in an iterative "online" fashion and as such, does not require a detailed modeling of power flows or line losses.

The main contribution of this paper is to describe a distributed algorithm for optimal dispatch of power generators, establish some of its optimality and convergence properties, illustrate its performance using simulation results and motivate further research into new techniques for the control and management of an electric grid with high penetration of alternative energy sources and advanced capabilities such as smart meters and flexible loads. Specifically, we show the following properties:

(A) For a constant load our algorithm exponentially drives the frequency deviation to zero, with or without power losses.
(B) Whenever there is a power imbalance, the algorithm reallocates the power generation across generators in such a way as to reduce the difference between the marginal costs of its constituent generators.
(C) If the load remains constant, under our algorithm, the network corrects for the frequency deviations and may reach an equilibrium without necessarily equalizing the marginal costs. *However, such a stationary point, with zero frequency deviation but unequal marginal costs is an unstable stationary point.* Taken together with (B), this implies that random load fluctuations will drive the algorithm eventually to equal marginal costs.
(D) If power losses are negligible, equal marginal costs implies optimality i.e. minimization of the total generation cost. Even with losses the algorithm continues to eventually equalize the marginal costs; however this is not optimal in general, in terms of minimization of total generation cost. We demonstrate that near optimality is achieved under small losses, and quantify the size of the small resulting suboptimality.

We also present simulation results to illustrate the above properties of the distributed algorithm and compare its performance with traditional centralized dispatch algorithms.

The rest of this paper is organized as follows. We present our fomulation of the dispatch problem and our mathematical model in Section II. Section III describes the distributed dispatch algorithm; some of the interesting optimality and convergence properties of the algorithm are derived in Section IV and Section V quantifies the level of suboptimality under small losses. The performance of the algorithm is compared with centralized dispatch methods and illustrated using numerical simulation results in Section VI. Section VII concludes.

II. Problem description

We model the economic dispatch problem as follows. We assume that there are N generators supplying power to the grid. At time-step k, we denote the total power consumed by $P_{load}[k]$, power losses in the grid by $P_{loss}[k]$, and the active power set point for generator i at the rated system frequency by $R_i[k],\ i \in 1\dots N$. As a result, the power imbalance in the system is given by

$$\Delta P[k] = P_{load}[k] - \sum_{i=1}^{N} R_i[k] + P_{loss}[k] \qquad \text{(II.1)}$$

This model is illustrated in Fig. 1. We neglect the effects of reactive power flows, voltage deviations and transients as is standard for economic dispatch problems.

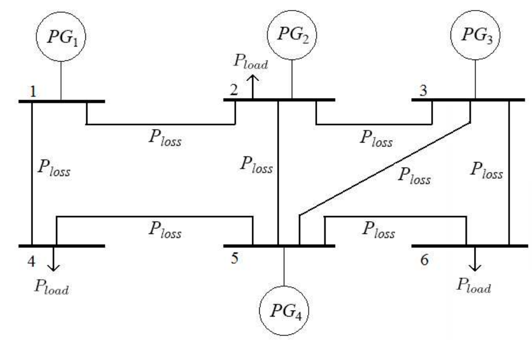

Fig. 1. Model of the electric grid for economic dispatch.

Note that $R_i[k]$ represents the active power *set-point*; the actual active power produced by each generator is determined by its primary controller which uses $R_i[k]$ as a reference. More precisely, we assume that each generator is equipped with a primary controller that implements a power-frequency characteristic (see for e.g. [25]) with a *negative droop*, so that the active power $P_i[k]$ produced at time k is related to the grid frequency $f[k]$ as:

$$P_i[k] = R_i[k] - \beta_i \Delta f[k] \qquad \text{(II.2)}$$
$$\text{where } \Delta f[k] \doteq f[k] - f_0$$

Note that at the rated frequency f_0 (usually 60 Hz or 50 Hz), $P_i[k] \equiv R_i[k]$. Since there is no energy storage in the grid, conservation of energy requires that

$$P_{load}[k] - \sum_{i=1}^{N} P_i[k] + P_{loss}[k] \equiv 0.$$

Combining (II.2) and (II.1), we get

$$\begin{aligned}\Delta P[k] &= P_{load}[k] - \sum_{i=1}^{N} P_i[k] - \Delta f[k] \sum_i \beta_i + P_{loss}[k] \\ &= -\beta \Delta f[k] \qquad \text{(II.3)}\end{aligned}$$

where we have denoted $\beta \doteq \sum_i \beta_i$. In other words, the total imbalance between the rated generation power and the load causes a proportional frequency deviation $\Delta f[k] = \frac{1}{\beta}\Delta P[k]$ on the grid that can be monitored continuously by each generator. This is analogous to the ACE observed by the secondary controller in a traditional LFC implementation. We assume that β remains constant for all values of $R_i[k]$ and $\Delta P[k]$. This is a reasonable assumption for small frequency deviations.

We now formally state the precise problem addressed in this paper.

Problem Statement: Let $J_i(P)$ be the cost function[2] for

[2] A time-varying cost function is one interesting way of accounting for the intermittency of alternative energy generators and is therefore of considerable interest. As long as the cost functions change over a time-scale significantly slower than the dynamics of the dispatch algorithm, the ideas in this paper can be applied to the time-varying case as well. For simplicity of presentation, we assume fixed cost functions in this paper.

generator i. The goal of the dispatch algorithm is to choose the $R_i[k]$ to force the power imbalance ΔP to zero and the marginal costs $J_i'(R_i)$ to be eventually all equal.

Effectively this minimizes the total cost $J = \sum_{i=1}^{N} J_i(R_i)$ subject to $\Delta P = 0$. The minimization is for the eventual cost rather than the aggregate incurred while achieving steady state. We assume that the cost functions $J_i(\cdot)$ satisfy certain mathematical properties that we describe next. Intuitively, these assumptions require that the cost function be monotonically increasing, convex and bounded. Let us denote the marginal costs as:

$$J_i'(P) \doteq \frac{dJ_i(P)}{dP}.$$

Assumption 2.1: Each $J_i(\cdot)$ is twice differentiable and strictly convex. Specifically, there exist $\eta_1 > 0$ and a positive nondecreasing function $f(\cdot)$, that is finite for all finite argument, such that for all P, and $i \in \{1, \cdots, N\}$, the second derivative

$$J_i''(P) \doteq \frac{d^2 J_i(P)}{dP^2},$$

satisfies $\eta_1 \leq J_i''(P) \leq f(P)$. Further $J_i''(\cdot)$ is piece-wise continuous.

We also assume non-zero idling cost:

Assumption 2.2: There exists $\eta_2 > 0$, such that $J_i'(0) > \eta_2$ for all $i \in \{1, \cdots, N\}$.

The above assumptions require that all the cost functions are twice differentiable with a positive piece-wise continuous second derivative. In practice, cost functions are often obtained heuristically and frequently modeled as a piece-wise linear function [26], [27] of the generated power. We note that the piece-wise linear cost function is just a convenient interpolation of a small number of known points on the cost curve [27]. They are useful mainly because it is possible to use linear programming techniques with such functions, though we are unaware of any algorithms that have a distributed implementation. It will be evident in the next section that the algorithm presented in this paper does not work for such functions as $J_i''(P)$ is zero or undefined for piece-wise linear functions, and J_i'' appears in the denominator of the power update kernel.

However, we note that as long as the marginal cost values at the tabulated points are positive and increase with increasing P_i, *one can just as easily interpolate them by a convex function,* and thus implement our distributed algorithm using these interpolated cost functions. Intuitively, these conditions (i.e. non decreasing slope of cost functions) simply mean that additional units of power become more expensive as generators operate closer to their maximum and are *usually satisfied by most practical cost functions. For example the points tabulated in Table 1 in [27] have all the properties that permit a convex positive slope interpolant to exist.*

At the same time a worthwhile line of future research is to explore whether, the algorithm of this paper can be modified to accommodate piecewise linear cost functions, that are monotonically increasing with non-decreasing slopes. We conjecture in Section IV that the algorithm in [1] can achieve power balance and minimum cost allocations for piecewise linear cost functions that have positive, increasing slope. Such an analysis will be a subject of future work.

Finally we assume that the power losses in the grid vary smoothly with the generator powers. We denote

$$\mathbf{R} \doteq [R_1,\ R_2,\ \ldots,\ R_N]^T \quad \text{(II.4)}$$

i.e. $\mathbf{R}(\cdot) : \mathbb{R} \to \mathbb{R}^N$ has elements representing the power set-points across the generators. Further we denote the power loss by

$$L(\mathbf{R}) \doteq P_{loss}. \quad \text{(II.5)}$$

Assumption 2.3: The function $L(\cdot)$ is nonnegative, differentiable and satisfies:

$$\gamma_i(\mathbf{R}) \doteq \frac{\partial L}{\partial R_i} \leq \gamma_0 < 1.$$

Thus, the γ_i, represent the fraction of an additional unit of power from generator i that is lost in the power grid.

Note that we do not assume any particular functional form for $L(\mathbf{R})$. *Nor do we assume that the total power loss can be written as a sum of individual loss terms that depend separately on the* R_i. Thus, for instance, Assumption 2.3 does not preclude the possibility that an increase in one of the generation powers R_i may actually decrease the total losses in the system i.e. $\gamma_i(\mathbf{R})$ can be negative (though we expect that this would be unusual in practice).

In addition to the above assumptions, for the purposes of the mathematical analysis in Section III, we also neglect the commonly imposed maximum and minimum limits on the power of the active generators. This assumption is not necessary for the working of the algorithm, however it makes the mathematical presentation considerably simpler.

It is easy to show using Lagrangian techniques that under Assumption 2.1, and zero power loss, the solution to the constrained optimization problem satisfies:

$$J_i'(R_i) \equiv \frac{dJ_i(R_i)}{dR_i} = \text{constant} \doteq \lambda, \forall i \in \{1 \ldots N\} \quad \text{(II.6)}$$

Equation (II.6) has the well known interpretation that at the minimum cost allocation of power, the marginal cost $J_i'(R_i)$ of an additional unit of power is constant across all generators. The optimal marginal cost is λ.

III. Distributed algorithm for optimal economic dispatch

We now describe our distributed algorithm. This is an iterative algorithm under which at time-step k, generator i updates its rated power as follows.

$$R_i[k+1] = \begin{cases} R_i[k] + \left(\frac{\alpha_1 \Delta P[k]}{J_i'(R_i[k]) J_i''(R_i[k])} \right), & \Delta P[k] \geq 0 \\ R_i[k] + \alpha_2 \Delta P[k] \frac{J_i'(R_i[k])}{J_i''(R_i[k])}, & \text{else} \end{cases} \quad \text{(III.7)}$$

where $\alpha_1 > 0$ and $\alpha_2 > 0$ are paramaters controlling the rate of adaptation. Note that all generators are calibrated with the same value for these parameters.

The intuition behind (III.7) is explained as follows. When the power imbalance $\Delta P[k]$ is positive, then the generators make a small increase to their rated powers in inverse proportion to their marginal cost. Thus generators with low marginal costs increase their allocation more rapidly than high cost generators. Conversely when the $\Delta P[k]$ is negative, then the low cost generator reduces its power less rapidly compared to high cost generators. The inclusion of the second derivative reflects the fact that a large second derivative causes larger changes to the marginal costs.

Observe that to implement this algorithm each generator only needs knowledge of its own cost function, in addition to a term proportional to the load imbalance that can be obtained by locally measuring the load frequency deviation. Thus the algorithm is implemented totally locally. As will be proved in the sequel, over time, this algorithm tends to equalize the marginal costs across generators and thus leads to the minimum cost solution, when $P_{loss}[k] \equiv 0$.

We next examine the properties of the algorithm in more detail.

IV. Properties of the Distributed Dispatch Algorithm

It is more convenient to analyze this algorithm by looking at its continuous time version described as follows. The results directly extend to the discrete time version for small α_i. The continuous time algorithm is as follows:

$$\frac{dR_i(t)}{dt} = \begin{cases} \alpha_1 \Delta P(t) \left(\frac{1}{J_i'(R_i(t)) J_i''(R_i(t))} \right), & \text{if } \Delta P(t) \geq 0 \\ \alpha_2 \Delta P(t) \frac{J_i'(R_i(t))}{J_i''(R_i(t))}, & \text{otherwise} \end{cases} \tag{IV.8}$$

Denote the vector corresponding to the optimal allocation (II.6) by $\mathbf{R}_{opt}$. Further let $\mathbf{1} \doteq [1,\ 1, \ldots,\ 1]^T$ denote the "all-ones" vector. Then we can write

$$\Delta P(t) \equiv P_{load}(t) - \mathbf{1}^T \mathbf{R}(t) + L(\mathbf{R}(t)) \tag{IV.9}$$

Observe, under Assumption 2.3, with

$$\Gamma(\mathbf{R}) = [\gamma_1(\mathbf{R}), \cdots, \gamma_N(\mathbf{R})]^T, \tag{IV.10}$$

with a constant load, we have:

$$\begin{aligned} \Delta \dot{P}(t) &= -\mathbf{1}^T \dot{\mathbf{R}}(t) + \dot{L}(\mathbf{R}(t)) \\ &= -[\mathbf{1} - \Gamma(\mathbf{R})]^T \dot{\mathbf{R}}(t). \end{aligned} \tag{IV.11}$$

Observe, *regardless of whether $\gamma_i(\mathbf{R})$ is negative*, because of Assumption 2.3, for all i the following holds:

$$1 - \gamma_i(\mathbf{R}) \geq 1 - \gamma_0 > 0. \tag{IV.12}$$

Consequently, with a positive $\Delta P(t)$, the $\dot{R}_i$ are strictly positive, and

$$\begin{aligned} \Delta \dot{P}(t) &= -[\mathbf{1} - \Gamma(\mathbf{R})]^T \dot{\mathbf{R}}(t) \\ &\leq -(1 - \gamma_0)\, \mathbf{1}^T \dot{\mathbf{R}} \\ &\leq 0. \end{aligned} \tag{IV.13}$$

Similarly, when $\Delta P(t)$ is negative, the $\dot{R}_i$ are nonpositive, and

$$\begin{aligned} \Delta \dot{P}(t) &= -[\mathbf{1} - \Gamma(\mathbf{R})]^T \dot{\mathbf{R}}(t) \\ &\geq -(1 - \gamma_0)\, \mathbf{1}^T \dot{\mathbf{R}} \\ &\geq 0. \end{aligned} \tag{IV.14}$$

We next prove the existence of a solution to (IV.8).

Theorem 4.1: Consider (IV.8), under assumptions 2.1, 2.2and 2.3, with $\mathbf{R}$ as in (II.4) and $P_{load}(t)$ a positive constant. Suppose for all $i \in \{1, \cdots N\}$, $R_i(0) > 0$, $\Delta P(0)$ is finite and the parameters $\alpha_1 > 0, \alpha_2 > 0$. Then the solution to (IV.8) exists and is unique. Further, for all $i \in \{1, \cdots, N\}$, and all $t \geq 0$, $J_i'(R_i(t)) > 0$ whenever $\Delta P(0) > 0$. On the other hand when $\Delta P(0) < 0$ for all $i \in \{1, \cdots, N\}$, and all $t \geq 0$, $J_i'(R_i(t)) \geq 0$.

Proof: Consider first the case where $\Delta P(0) > 0$. Then the first clause in (IV.8) applies at least until at some $T > 0$, $\Delta P(T) = 0$. In the interval $t \in [0, T)$, (IV.8) becomes:

$$\frac{d}{dt} \{J_i'(R_i(t))\}^2 = 2\alpha_1 \Delta P(t). \tag{IV.15}$$

Observe while $J_i'(R_i(t)) \geq 0$, $\{J_i'(R_i(t))\}^2$ uniquely specifies $J_i'(R_i(t))$, which in turn, because of (2.1) uniquely specifies $R_i(t)$ and hence $\Delta P(t)$. Thus as $\Delta P(0)$ is finite, at least for some $\epsilon > 0$, and $t \in [0, \epsilon)$, a unique solution exists, and is in fact continuous.

Further for all $t \in [0, \epsilon)$, the $R_i(t)$ are strictly increasing, as because of Assumption 2.1 are the $J_i'(R_i(t))$. Further because of (IV.13), $\Delta P(t)$ is decreasing and is hence bounded by $\Delta P(0)$. Thus extending this argument for $t = \epsilon$ and beyond, a unique continuous solution exists for all $t \in [0, T)$. Then the existence and uniqueness of the solution is guaranteed as by definition $\Delta P(T) = 0$ and $\Delta P = 0$ is a stationary point of the algorithm.

Now suppose $\Delta P(0) < 0$. Then the second clause in (IV.8) applies at least until at some $T > 0$, $\Delta P(T) = 0$. In the interval $t \in [0, T)$, (IV.8) becomes:

$$\frac{d}{dt} \{J_i'(R_i(t))\} = \alpha_2 J_i'(R_i(t)) \Delta P(t). \tag{IV.16}$$

Observe because of Assumption 2.1, $J_i'(R_i(t))$ uniquely specifies $R_i(t)$ and hence $\Delta P(t)$. Thus at least for some $\epsilon > 0$, and $t \in [0, \epsilon)$, a unique solution exists, and is again continuous. Further for all $t \in [0, \epsilon)$, the $R_i(t)$ are strictly deccreasing as are the $J_i'(R_i(t))$. Further, because of Assumption 2.3 and (IV.11), $\Delta P(t)$ is increasing on this interval and hence has magnitude bounded by $|\Delta P(0)|$. Thus arguing as above a unique continuous solution exists for all $t \in [0, T)$. Then the existence and uniqueness of the solution is again guaranteed as $\Delta P = 0$ is a stationary point of the algorithm.

The case of $\Delta P(0) = 0$ is trivial, because it represents a stationary point of the algorithm.

Finally, if $\Delta P(0) > 0$, the $R_i(t)$ and hence the $J_i'(R_i(t))$ are nondecreasing. Consequently, under assumption (2.2), $J_i'(R_i(t)) > 0$ for all t. On the other hand if $\Delta P(0) < 0$, (IV.16) holds. Observe, $J_i'(R_i(t)) = 0$ is a stationary point of (IV.16). Thus, as $J_i'(R_i(0)) > 0$ and the solution is continuous, $J_i'(R_i(t)) \geq 0$ for all t.

■

We next show that (IV.8) induces $\Delta P(t)$ to converge exponentially to zero.

Theorem 4.2: Under the conditions of Theorem 4.1 suppose $P_{load}(t)$ is a positive constant. Then the power imbalance $\Delta P(t)$ converges exponentially to zero.

Proof: Let us first consider the case where the initial power imbalance $\Delta P(0) > 0$. Then it necessarily follows that $\Delta P(t) \geq 0$, $\forall t \geq 0$ i.e. that the power imbalance is always non-negative. It also follows from (IV.8) that $R_i(t)$, $\forall i$ are monotonically non-decreasing functions of time. Thus $R_i(t) \geq R_i(0)$ and $J_i(R_i(t)) \geq J_i(R_i(0))$ for all $t \geq 0$. Therefore, using (IV.13) we get:

$$\frac{d\Delta P(t)}{dt} \leq -(1-\gamma_0)\,\mathbf{1}^T\dot{\mathbf{R}}(t) = -\alpha_1(1-\gamma_0)\,\Delta P(t)\sum_{i=1}^{N}\frac{1}{J_i'(R_i(t))J_i''(R_i(t))}. \quad \text{(IV.17)}$$

We now assert that for some $\mu_1 > 0$,

$$\frac{d\Delta P(t)}{dt} \leq -\mu_1\Delta P(t). \quad \text{(IV.18)}$$

To establish a contradiction suppose (IV.18) is false. As from (IV.8) $R_i(t)$ are positive and increasing, because of Assumption 2.1 for every $\epsilon > 0$, there exists a t_1 such that

$$0 < \sum_{i=1}^{N}\frac{1}{J_i'(R_i(t_1))J_i''(R_i(t_1))} \leq \epsilon. \quad \text{(IV.19)}$$

Because of Assumption 2.1 this must mean that for every $M_1 > 0$ there is a t_1 such that

$$\max_{i\in\{1,\cdots,N\}} J_i'(R_i(t_1))J_i''(R_i(t_1)) > M_1.$$

Thus again from Assumption 2.1 for every $M_2 > 0$ there is a t_1 such that

$$\max_{i\in\{1,\cdots,N\}} R_i(t_1) > M_2.$$

Choose $M_2 = P_{load}$. As all $R_i(t) > 0$, $\Delta P(t_1) < 0$. This is impossible as, from Theorem 4.1, all trajectories are continuous, $\Delta P(0) > 0$ and $\Delta P = 0$ is a stationary point of (IV.8). Thus indeed (IV.18) holds. Consequently, $|\Delta P(t)| \leq \Delta P(0)\exp(-\mu_1 t)$ which is the desired result.

Now suppose that $\Delta P(0) < 0$. Then for all $t \geq 0$, $\Delta P(t) \leq 0$. Thus for all $t \geq 0$,

$$\sum_{i=1}^{N} R_i(t) \geq P_{load} > 0. \quad \text{(IV.20)}$$

Thus,

$$R^* = \max_{i\in\{1,\cdots,N\}}\{R_i(0)\} > 0,$$

and at every t, there is at least on i, for which $R_i(t) > 0$. From Assumption 2.2 this implies that for every t, there exists an i, such that $J_i'(R_i(t)) > 0$. Further because of Theorem 4.1, for all $t \geq 0$ and all $i \in \{1,\cdots,N\}$, $J_i'(R_i(t)) \geq 0$. Then, because of Assumptions 2.1 and 2.2, for all $t \geq 0$, we have

$$\sum_{i=1}^{N}\frac{J_i'(R_i(t))}{J_i''(R_i(t))} \geq \frac{\eta_2}{f(R^*)} = \mu_2 > 0. \quad \text{(IV.21)}$$

Therefore, since $\Delta P(t) < 0$, using (IV.14) we obtain:

$$\begin{aligned}\frac{d\Delta P(t)}{dt} &\geq -(1-\gamma_0)\mathbf{1}^T\dot{\mathbf{R}}(t) \\ &= -(1-\gamma_0)\alpha_2\Delta P(t)\sum_{i=1}^{N}\frac{J_i'(R_i(t))}{J_i''(R_i(t))} \\ &\geq -\mu_2\alpha_2(1-\gamma_0)\Delta P(t) \quad \text{(IV.22)}\end{aligned}$$

and the result follows again as $\Delta P(t) < 0$. ■

Note that Theorem 4.2 guarantees that the algorithm reaches an equilibrium point exponentially fast. For technical facility, the proof of the theorem uses bounds in e.g. (IV.21) that are conservative and understate the rate of convergence. This result can be trivially extended to show that when $P_{load}(t)$ is not constant, then ultimately the load deficit will be proportional to the rate of change of the load and inversely proportional to constants like μ_2, i.e. slowly varying loads are well tracked.

The Theorem does not guarantee that the equilibrium point has minimum cost in the lossless case. Indeed for a constant load, $\Delta P(t)$ may converge before the marginals are equalized. The next theorem shows that in fact the algorithm drives the marginals towards equalization *while a load imbalance persists*.

Theorem 4.3: Under the conditions of Theorem 4.1, suppose $\Delta P(t) \neq 0$, and for some i, j, $J_i'(R_i(t)) > J_j'(R_j(t))$. Then

$$\left(\frac{d(J_i'(R_i(t)) - J_j'(R_j(t)))}{dt}\right) < 0.$$

Proof: Once again consider first the case of $\Delta P(0) > 0$. Then as $\Delta P(0) \geq 0$ for all $t \geq 0$, we have that

$$\begin{aligned}&\frac{d(J_i'(R_i(t)) - J_j'(R_j(t)))}{dt} \\ &= J_i''(R_i(t))\frac{dR_i(t)}{dt} - J_j''(R_j(t))\frac{dR_j(t)}{dt} \\ &= \alpha_1\Delta P(t)\left(\frac{1}{J_i'(R_i(t))} - \frac{1}{J_j'(R_j(t))}\right) \\ &= \frac{\alpha_1\Delta P(t)}{J_i'(R_i(t))J_j'(R_j(t))}\left(J_j'(R_j(t)) - J_i'(R_i(t))\right) \\ &< 0, \quad \text{(IV.23)}\end{aligned}$$

where the last inequality exploits the fact that because of Theorem 4.1 the marginals are always positive.

On the other hand, when $\Delta P(0) < 0$, as $\Delta P(t) \leq 0$ for all $t \geq 0$, there holds:

$$\begin{aligned}&\frac{d(J_i'(R_i(t)) - J_j'(R_j(t)))}{dt} \\ &= J_i''(R_i(t))\frac{dR_i(t)}{dt} - J_j''(R_j(t))\frac{dR_j(t)}{dt} \\ &= \alpha_2\Delta P(t)\left(J_j'(R_j(t)) - J_i'(R_i(t))\right) \\ &< 0. \quad \text{(IV.24)}\end{aligned}$$

■

To complete our study of equalization of marginals we next present the following theorem.

Theorem 4.4: Assume the conditions of Theorem 4.1hold.

(a) Consider the stationary point where $\Delta P(t) = 0$, but for some $\{i,j\} \subset \{1,\cdots,N\}$,

$$J_i'(R_i(t)) \neq J_j'(R_j(t)).$$

(b) Suppose there exists a stationary point such that $\Delta P(t) = 0$, and for all $\{i,j\} \subset \{1,\cdots,N\}$,

$$J_i'(R_i(t)) = J_j'(R_j(t)).$$

Then this stationary point is unique and locally stable.

Proof: The convexity of the $J_i(\cdot)$ ensures that there is a one to one mapping between $J_i'(R_i)$ and R_i. Suppose $\mathbf{R}^*$ is the stationary point in (a). Then for every $\epsilon > 0$ there is a perturbation

$$\delta\mathbf{R} = [\delta R_1, \cdots, \delta R_N]^T,$$

and $\mathbf{R}(0) = \mathbf{R}^* + \delta\mathbf{R}$, such that $\|\delta\mathbf{R}\| < \epsilon$ has the property that for each pair i,j,

$$\left|J_i'(R_i + \delta R_i) - J_j'(R_j + \delta R_j)\right| \le \left|J_i'(R_i) - J_j'(R_j)\right|$$

with strict inequality for at least one i,j pair. Then from convexity and Theorem 4.3, the trajectory moves further away from $\mathbf{R}$. Hence from [28] this stationary point is locally unstable.

Now consider a stationary point as in (b). To establish a contradiction suppose there are two such stationary points $\mathbf{R}_1$ and $\mathbf{R}_2$. Then atleast one element of $\mathbf{R}_1$ is larger than its counterpart in $\mathbf{R}_2$, and another smaller. Then convexity precludes the possibility of equal marginal costs.

Define the Lyapunov function:

$$V(\mathbf{R}) = \sum_{i=1}^{N} \sum_{j\in\{1,\cdots,N\}\{i\}} \left(J_i'(R_i) - J'(R_j)\right)^2 + (\Delta P(t))^2.$$

Observe $V(\mathbf{R}) \ge 0$ with equality iff $\mathbf{R}$ is the unique stationary point defined in (b). Then from Theorem 4.4, and (IV.8) the $\dot{V} \le 0$. Hence from [28] the result follows. ■

Taken together, the significance of Theorems 4.3 and 4.4, is as follows. While $\Delta P(t) \neq 0$, the algorithm will tend to drive the marginals closer. If $\Delta P(t)$, becomes zero before the marginals are equalized, then the slightest noise in the R_i or load fluctuations that enforce the condition $\Delta P(t) \neq 0$, will again tend to drive the marginals closer to each other. Over time the practical effect of this is to equalize the marginals.

Observe that all results in this section accommodate power losses. Of course equalization of the marginals is suboptimal unless the power losses are zero. Section V quantifies the level of suboptimality under small power losses.

Remark. In view of the analysis presented above, we can rewrite the update rule (IV.8) as

$$\begin{aligned} \frac{dJ_i'(R_i(t))}{dt} &= \Phi(R_i(t))\Delta P(t) \\ \text{where } \Phi(R) &= \begin{cases} \frac{\alpha_1}{J_i'(R)}, & \text{if } \Delta P(t) \ge 0 \\ \alpha_2 J_i'(R), & \text{otherwise} \end{cases} \end{aligned}$$

This is reminiscent of Lagrangian algorithms such as lambda iteration, that iteratively "search through" the space of marginal costs until the power imbalance is removed. Indeed, by changing the step-size rule $\Phi(R)$, we can design an entire family of distributed Lagrangian algorithms for the dispatch problem. However, this family of algorithms can vary widely in their convergence and stability properties and their analysis is beyond our scope here. A more detailed exploration of such algorithms is an important topic for future work.

We conclude this section by briefly addressing the issue of piecewise linear cost functions with increasing slopes. In this case second derivatives of all cost functions are equal, and in fact zero, almost everywhere. Thus the weighting by the reciprocal of the second derivative in (III.7) is not only infeasible, but also unnecessary. Thus the algorithm of [1] described below, duly modified to deal with points at which the cost functions are non-differentiable, is a potential candidate:

$$R_i[k+1] = \begin{cases} R_i[k] + \left(\frac{\alpha_1 \Delta P[k]}{J_i'(R_i[k])}\right), & \Delta P[k] \ge 0 \\ R_i[k] + \alpha_2 \Delta P[k] J_i'(R_i[k]), & \text{else} \end{cases} \tag{IV.25}$$

The continuous time counterpart of this algorithm is:

$$\frac{dR_i(t)}{dt} = \begin{cases} \alpha_1 \Delta P(t)\left(\frac{1}{J_i'(R_i(t))}\right), & \text{if } \Delta P(t) \ge 0 \\ \alpha_2 \Delta P(t) J_i'(R_i(t)), & \text{otherwise} \end{cases} \tag{IV.26}$$

It is readily seen that as long the marginal costs are all positive, the argument in the proof of Theorem 4.2 goes through, and $\Delta P(t)$ converges exponentially to zero. As for the counterpart of Theorem 4.3, let us consider the case where $\Delta P(t) > 0$, the argument for negative $\Delta P(t)$ being very similar. It is readily seen that:

$$J_i'(R_i(t)) > J_k'(R_k(t)) \Rightarrow \frac{dR_i(t)}{dt} < \frac{dR_k(t)}{dt}. \tag{IV.27}$$

Hence as the cost functions are piecewise linear and have increasing slopes, a simple argument shows that the marginal costs must tend to approach each other. Actual equalization will, however depend on whether there are operating points at which all marginal costs are equal, and whether, these operating points are compatible with the load requrements. Indeed in the piecewise linear case equal marginal costs is not necessary for optimality. We conjecture however, that (IV.27) leads to eventual optimality in the case where the cost functions are piecewise linear with positive increasing slopes.

V. Lossy Performance

Thus, with zero losses the algorithm eventually achieves optimum performance. We now quantify the level of suboptimality when γ_0 in Assumption 2.3 is nonzero but small. This is physically reasonable because, recall that the γ_i (in the sequel and here we drop the argument $\mathbf{R}$), represents the fraction of an additional unit of power that is lost in the power grid, and in any well-designed grid, we expect this to be very small e.g. $\gamma_0 < 10\%$. In this section we demonstrate that for small γ_0, the level of suboptimality is quadratic in γ_0.

We note that Theorems 4.2 and 4.3 continue to hold for the algorithm defined in (III.7) even with power losses, however, it is no longer true that the equal marginal cost allocation achieves the minimum cost. Indeed, we can show using Lagrangian methods that the minimum cost allocation

$\mathbf{R}_{opt} \doteq [R_{1,opt},\ R_{2,opt},\ \ldots,\ R_{N,opt}]^T$ satisfies

$$\left(\frac{1}{(1-\gamma_i)}\frac{dJ_i}{dR_i}\right)_{R_i=R_{i,opt}} = \left(\frac{1}{(1-\gamma_j)}\frac{dJ_j}{dR_j}\right)_{R_j=R_{j,opt}},\ \forall\ i,j \tag{V.28}$$

We next show that the resulting sub-optimality is small provided that the marginal power losses (i.e. γ_i) are not too large.

We have noted that the algorithm of (III.7) tends to equalize marginal cost allocation which may not be the same as (V.28). Let us denote this equalized marginal cost allocation by $\mathbf{R}_0 \doteq [R_{1,0},\ R_{2,0},\ \ldots,\ R_{N,0}]^T$ and the marginal cost corresponding to this allocation as

$$\lambda_0 \doteq \left(\frac{dJ_i}{dR_i}\right)_{R_i=R_{i,0}} \equiv \left(\frac{dJ_j}{dR_j}\right)_{R_j=R_{j,0}}.$$

From our assumption that the marginal power losses are small, under suitable smoothness assumptions, $\mathbf{r} \doteq \mathbf{R}_{opt} - \mathbf{R}_0$ will be small, where $\mathbf{r} \equiv [r_1,\ r_2,\ \ldots,\ r_N]^T$ is the deviation of the equal marginal cost allocation from the minimum cost allocation. Thus we can write

$$\begin{aligned}\left(\frac{dJ_i}{dR_i}\right)_{R_i=R_{i,opt}} &\approx \left(\frac{dJ_i}{dR_i}\right)_{R_i=R_{i,0}} + r_i\frac{d^2J_i}{dR_i^2}\\ &= \lambda_0\left(1+\frac{r_i}{\lambda_0}\frac{d^2J_i}{dR_i^2}\right)\end{aligned} \tag{V.29}$$

Furthermore, since both $\mathbf{R}_0$ and $\mathbf{R}_{opt}$ satisfy the power balance constraint, we have $\mathbf{1}^T\mathbf{R}_{opt} - L(\mathbf{R}_{opt}) \equiv \mathbf{1}^T\mathbf{R}_0 - L(\mathbf{R}_0) \equiv P_{load}$. Thus we have $\mathbf{1}^T\mathbf{r} = L(\mathbf{R}_{opt}) - L(\mathbf{R}_0) \equiv \sum_i \gamma_i r_i$, which gives after rearrangement $\sum_i(1-\gamma_i)r_i = 0$. Thus, it is not possible to have either $r_i \geq 0, \forall i$ or $r_i \leq 0, \forall i$. Without loss of generality, through a relabeling of generators if need be, assume that $r_1 > 0$ and $r_2 < 0$. Using (V.28) and (V.29), we have

$$\begin{aligned}\frac{1}{(1-\gamma_1)}\lambda_0\left(1+\frac{r_1}{\lambda_0}\frac{d^2J_1}{dR_1^2}\right) &\approx \frac{1}{(1-\gamma_2)}\lambda_0\left(1+\frac{r_2}{\lambda_0}\frac{d^2J_2}{dR_2^2}\right)\\ &\leq \frac{\lambda_0}{(1-\gamma_0)}\end{aligned} \tag{V.30}$$

where we have used the fact that $\gamma_2 \leq \gamma_0$ and the assumption that $r_2 < 0$. Thus using the fact that $\gamma_1 < \gamma_0$, there holds:

$$r_1 \leq \frac{\lambda_0\gamma_0}{(1-\gamma_0)}\frac{1}{\frac{d^2J_1}{dR_1^2}} \leq \frac{\lambda_0\gamma_0}{\eta_1(1-\gamma_0)} \tag{V.31}$$

Using the same argument for all generators i, we have

$$|r_i| \leq \frac{\gamma_0}{(1-\gamma_0)}\frac{\lambda_0}{\eta_1} \tag{V.32}$$

Finally we have

$$\begin{aligned}J(\mathbf{R}_{opt}) &\equiv \sum_i J_i(R_{i,opt})\\ &\approx J(\mathbf{R}_0) + \sum_i r_i\left(\frac{dJ_i}{dR_{i,0}}\right)\\ &= J(\mathbf{R}_0) + \lambda_0\sum_i r_i\\ &= J(\mathbf{R}_0) + \lambda_0\left(L(\mathbf{R}_0) - L(\mathbf{R}_{opt})\right) \qquad \text{(V.33)}\\ &\geq J(\mathbf{R}_0) - \lambda_0\gamma_0\sum_i |r_i|\\ &\geq J(\mathbf{R}_0) - \frac{N\gamma_0^2}{(1-\gamma_0)}\frac{\lambda_0^2}{\eta_1} \qquad \text{(V.34)}\end{aligned}$$

Equation (V.34) shows that the suboptimality of the equal marginal cost solution varies with γ_0^2 and therefore is small provided the power loss is not too large. This observation is verified by numerical simulation in Section VI.

VI. Simulation Results and gain selection

We begin with some comments on the selection of the gain parameters α_i. These parameters affect only the rate of convergence of the continuous time version of this algorithm, rather than whether or not convergence occurs at all. In the discrete time case of (III.7), however, too large a value of α_i may destablize.

To be specific consider the case where $\Delta P > 0$, as similar considerations affect the case where $\Delta P < 0$. In this case without losses and with constant P_{load} there holds:

$$\begin{aligned}\Delta P(t+1) &= P_{load} - \mathbf{1}^T\mathbf{R}(t+1)\\ &= P_{load} - \mathbf{1}^T\mathbf{R}(t) - \mathbf{1}^T\left(\mathbf{R}(t+1) - \mathbf{R}(t)\right)\\ &= \Delta P(t) - \alpha_1\Delta P(t)\sum_{i=1}^N \frac{1}{J_i'(R_i(t))J_i''(R_i(t))}\\ &\leq (1-\mu_1)\Delta P(t),\end{aligned}$$

where μ_1 is as defined in the proof of Theorem 4.2. Thus at the minimum one must keep α_1 small enough to ensure that $0 < \mu_1 < 1$. This would require rough *a priori* estimates of bounds on the initial marginal costs and their derivatives. Thus an initial centralized intervention is needed to set the α_i accordingly. Beyond this several other considerations intervene. First, very small values of α_i will cause slow convergence. Second large values will cause poor noise performance. Third, a stable network can be destabilized by large power swings. Equally, it is a standard conclusion from stability theory, exploited extensively in the adaptive systems literature, [29], that sufficiently small fluctuations, ensured in this case by sufficiently small α_i, will leave an otherwise stable network stable. Finally they must be kept small enough so that the individual generator dynamics can cope with the demands of the algorithm. To summarize, the α_i must be chosen on a case by case basis, according to the bounds, albeit rough, on the anticipated loads and starting initial conditions, and on the knowledge of the network and generator dynamics, and the speed with which demands must be met. *At the same time, once the network has been fixed, this is a once only disign,* *that need not be changed absent substantial changes in the network conditions.*

We now present numerical results to provide some intuition into the performance of the algorithm. The α_i were selected in these simulations using all the considerations note above barring the ones associated with network and generator dynamics. We consider the dispatch problem considered in [13] (Example 7.4) of allocating 1800 kW of total generation in a small grid with 6 generators. Each generator has a quadratic cost function i.e. generator i has a cost given by $J_i(R) \equiv c_i R^2 + b_i R + a_i$, where $c_i > 0$ guarantees the convexity of the cost function.

We simulated this system starting from an arbitrarily chosen, suboptimal allocation among the generators. The total load power fluctuates ramdomly around an average value of 1800 kW; these fluctuations cause power imbalances, and the resulting frequency deviations are used by the individual generators to adjust their power set-points according to (III.7). Note that the load fluctuations are essential for creating the temporary frequency deviations that drive the overall allocation towards its optimum values; however, in practice, the same effect can be obtained by deliberately introducing random fluctuations into the generation power itself if desired. The iteration time for the algorithm is taken to be 10 seconds. In this case $\alpha_1 = 0.0215$ and $\alpha_2 = 0.000075$.

Figure 2 shows the individual generation powers as well as the total cost as a function of time according to the distributed algorithm. Also shown is the minimum cost obtained by a numerical method based on neural network computing in [13]. We note that the algorithm matches within a few minutes the cost achieved by the centralized method in [13] (which was also verified by comparing with the lambda iteration method).

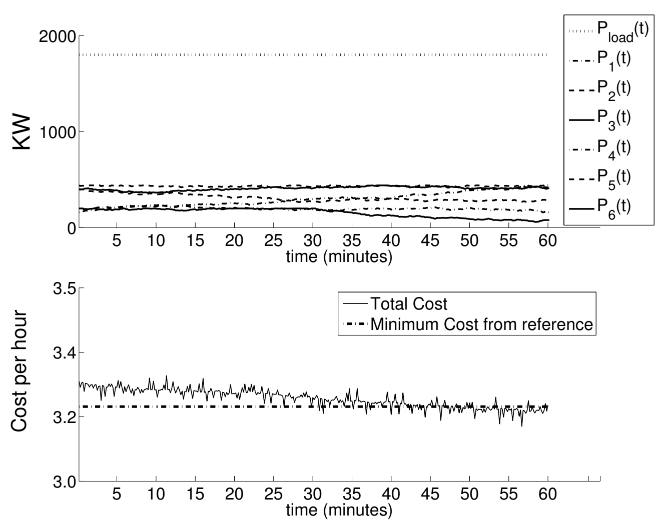

Fig. 2. Performance of the distributed dispatch algorithm.

Figure 3 shows another example of a dispatch problem, this time there are power losses in the grid that amounts to approximately $1-2\%$ of the total generated power. This problem is solved using a numerical particle-swarm optimization technique in [14]. While our distributed algorithm does not specifically account for power losses, we expect from the analysis of Section V that for the small levels of losses, the algorithm should achieve near-optimal performance, and we can indeed see from Fig. 3 that it eventually matches the minimum cost achieved by the centralized optimization algorithm in [14]. In this case, $\alpha_1 = 0.06$ and $\alpha_2 = 0.001$.

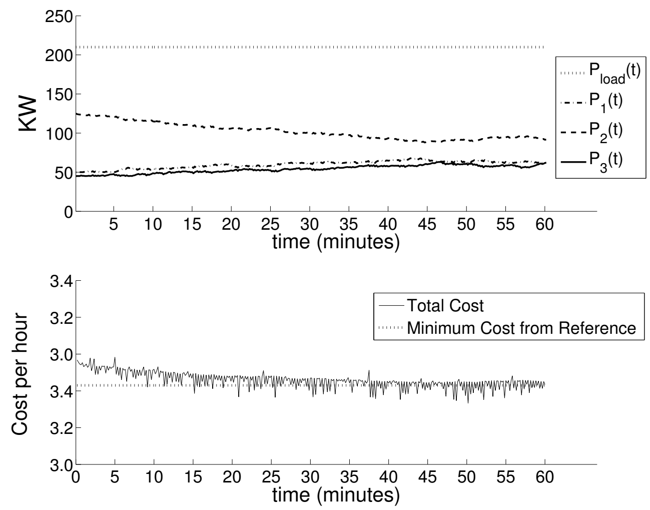

Fig. 3. The distributed dispatch algorithm in a network with power losses.

Figure 4 shows a simulation of a 3 generator system [12], this time showing that the distributed algorithm effectively reduces a large initial power imbalance of 100 kW to zero within a small number of iterations. In practice, such a large power imbalance may arise if a generator goes offline because of a fault; significant transients can be expected under such conditions, and we make no claim that Fig. 4 represents an accurate model of the state of the generators in such a case. Instead, this plot is intended to illustrate the fast exponential convergence response property of the algorithm s shown in Theorem 4.2. The parameter values for this simulation were $\alpha_1 = 0.031$ and $\alpha_2 = 0.000475$.

Finally, we also simulated a 20 generator system to show that the distributed algorithm scales well to larger systems. This simulation is based on the dispatch problem described in example 2 of [30] with one modification: we scaled up the quadratic term (parameters c_i in Table 2 of [30]) so that the marginal costs of the generators vary over a larger range and the performance of the algorithm is easier to visualize. This example also has minimum and maximum constraints on the power allocated to each generator, and thus also shows that such constraints are easily handled by the distributed algorithm. In this simulation, we used $\alpha_1 = 0.45,\ \alpha_2 = 0.0007$.

The simulation results are shown in Fig. 5; we note that as expected the marginal costs of the generators become more equal over time. However, there are power losses in this example and therefore as discussed earlier the equal marginal cost condition only approximately achieves the minimum cost.

VII. Conclusion

We proposed a new distributed approach to economic dispatch, where each generator independently adjusts its power in

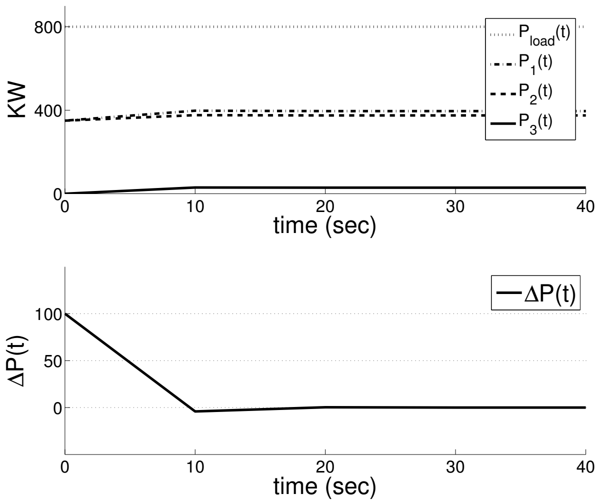

Fig. 4. Exponential convergence of power imbalance.

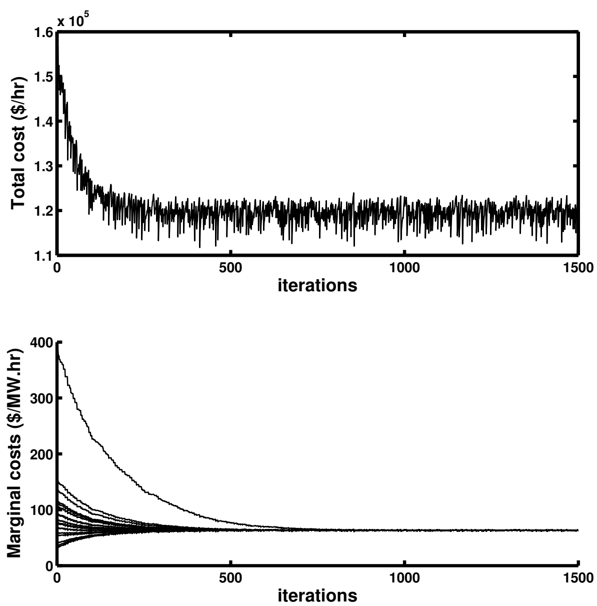

Fig. 5. Total cost and marginal costs in a 20 generator system.

response to frequency deviations on the grid, and we showed that it is possible to achieve the minimum cost allocation using such a method. We demonstrated through analysis and numerical simulations the effectiveness of the distributed approach and argued that it is especially attractive to an electric grid with distributed generation, smart metering capabilities and alternative energy sources.

This opens up many interesting issues for future work. While we focused on the economic dispatch problem in this paper, it is also interesting to explore a distributed approach to other problems such as reactive power control. Also while the algorithm studied in this paper only uses measurements of frequency deviations on the grid, the future electric grid may offer communications infrastructure that provides much more detailed information on the state of the grid (e.g. real-time power flow measurements), and it is an interesting open problem to develop techniques to use such information to optimize the performance of the grid. Similarly, it would also be interesting to consider how to best take advantage of energy storage technologies to increase the efficiency of the electric grid.

As noted in this paper some times one interpolates tabular data to formulate piecewise linear cost functions that have positive increasing derivatives. As noted in the paper, our algorithm does not work in such cases. However, as also noted in the paper such data can with facility be also interpolated by convex cost functions with positive strictly increasing derivatives to which our algorithm can be applied. Nonetheless we have conjectured that our algorithm from [1] should work with piecewise linear cost functions that have positive increasing derivatives. A closer examination of this question is another interesting topic for future work.

We observed earlier that our algorithm can be thought of as an iterative implementation of Lagrangian optimization. Exploring variations of this algorithm offering a variety of convergence and stability properties is another interesting area for future work.

Finally, this paper does not take network and generator dynamics explicitly into account. Another related open problem is to design controllers for wind turbine and other alternative energy generators that allow flexible real-time control over the voltage, active and reactive power of the generators.

References

[1] R. Mudumbai, S. Dasgupta, and B. Cho, "Distributed control for optimal economic dispatch of power generators," in *Proceedings of the Chinese Control Conference*, 2010.

[2] S. Marvin, H. Chappells, and S. Guy, "Pathways of smart metering development: shaping environmental innovation," *Computers, Environment and Urban Systems*, vol. 23, no. 2, pp. 109–126, 1999.

[3] D. Milborrow, "Penalties for intermittent sources of energy," *Cabinet Office, London*, p. 17, 2001.

[4] C. Marnay and G. Venkataramanan, "Microgrids in the evolving electricity generation and delivery infrastructure," 2006, p. 5.

[5] R. Dugan and T. McDermott, "Distributed generation," *IEEE Industry Applications Magazine*, vol. 8, no. 2, pp. 19–25, Mar/Apr 2002.

[6] M. Amin and B. Wollenberg, "Toward a smart grid: power delivery for the 21st century," *IEEE Power and Energy Magazine*, vol. 3, no. 5, pp. 34–41, 2005.

[7] N. Jaleeli, L. VanSlyck, D. Ewart, L. Fink, and A. Hoffmann, "Understanding automatic generation control," *Power Systems, IEEE Transactions on*, vol. 7, no. 3, pp. 1106–1122, Aug 1992.

[8] K. Tomsovic, D. Bakken, V. Venkatasubramanian, and A. Bose, "Designing the next generation of real-time control, communication, and computations for large power systems," *Proceedings of the IEEE*, vol. 93, no. 5, pp. 965–979, 2005.

[9] Y. G. Rebours, D. S. Kirschen, M. Trotignon, and S. Rossignol, "A survey of frequency and voltage control ancillary services part i: Technical features," *IEEE Transactions on Power Systems*, vol. 22, no. 1, pp. 350–357, Feb. 2007.

[10] F. Wu, K. Moslehi, and A. Bose, "Power system control centers: past, present, and future," *Proceedings of the IEEE*, vol. 93, no. 11, pp. 1890–1908, 2005.

[11] B. Chowdhury and S. Rahman, "A review of recent advances in economic dispatch," *Power Systems, IEEE Transactions on*, vol. 5, no. 4, pp. 1248–1259, Nov 1990.

[12] H. Saadat, *Power system analysis*. WCB/McGraw-Hill, Boston, 1999.

[13] A. Mohammadi, M. Varahram, and I. Kheirizad, "Online solving of economic dispatch problem using neural network approach and comparing it with classical method," in *Emerging Technologies, 2006. ICET'06. International Conference on*. IEEE, 2007, pp. 581–586.

[14] C. Chen, "Economic dispatch using simplified personal best oriented particle swarm optimizer," in *Electric Utility Deregulation and Restructuring and Power Technologies, 2008. DRPT 2008. Third International Conference on*. IEEE, 2008, pp. 572–576.

[15] F. Schweppe, R. Tabors, J. Kirtley, H. Outhred, F. Pickel, and A. Cox, "Homeostatic utility control," *Power Apparatus and Systems, IEEE Transactions on*, vol. PAS-99, no. 3, pp. 1151 –1163, May 1980.

[16] F. Alvarado, "Controlling power systems with price signals," *Decision Support Systems*, vol. 40, no. 3-4, pp. 495–504, 2005.

[17] A. Jokic, P. Van den Bosch, and M. Lazar, "Distributed price-based optimal control of power systems," in *IEEE International Conference on Control Applications*, 2007, pp. 910–915.

[18] D. P. Bertsekas, *Nonlinear Programming*, 2nd ed. Athena Scientific, Sep. 1999.

[19] G. Miranda, "Be prepared! [power industry deregulation]," *IEEE Industry Applications Magazine*, vol. 9, no. 2, pp. 12 – 20, mar. 2003.

[20] R. Gilbert and E. Kahn, *International comparisons of electricity regulation*. Cambridge Univ Press, 1996.

[21] D. Das and B. Wollenberg, "Risk assessment of generators bidding in day-ahead market," *IEEE Transactions on power systems*, vol. 20, no. 1, pp. 416–424, 2005.

[22] A. Sakis Meliopoulos, G. Cokkinides, and A. Bakirtzis, "Load-frequency control service in a deregulated environment," vol. 3, 1998, pp. 24 –31.

[23] U. S. D. of Energy, "Economic dispatch of electric generation capacity," *A REPORT TO CONGRESS AND THE STATES*, 2007.

[24] L. Vargas, V. Quintana, and A. Vannelli, "A tutorial description of an interior point method and its applications to security-constrained economic dispatch," *Power Systems, IEEE Transactions on*, vol. 8, no. 3, pp. 1315–1324, 2002.

[25] A. Engler, "Applicability of droops in low voltage grids," *International Journal of Distributed Energy Resources*, vol. 1, no. 1, pp. 3–15, 2005.

[26] X. Han, H. Gooi, and D. Kirschen, "Dynamic economic dispatch: feasible and optimal solutions," *IEEE Transactions on Power Systems*, vol. 16, no. 1, pp. 22–28, 2002.

[27] M. Aganagic and S. Mokhtari, "Security constrained economic dispatch using nonlinear dantzig-wolfe decomposition," *Power Systems, IEEE Transactions on*, vol. 12, no. 1, pp. 105 –112, Feb. 1997.

[28] H. Khalil and J. Grizzle, *Nonlinear systems*. Prentice hall Englewood Cliffs, NJ, 2002.

[29] B. D. O. Anderson, R. R. Bitmead, C. R. Johnson, P. V. Kokotovic, R. L. Kosut, I. M. Y. Mareels, L. Praly, and B. Riedle, *Stability of Adaptive Systems*. MIT Press, 1986.

[30] C.-T. Su and C.-T. Lin, "New approach with a hopfield modeling framework to economic dispatch," *IEEE Transactions on Power Systems,*, vol. 15, no. 2, pp. 541 –545, may 2000.

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